

Notes on trying to obtain a simple alternative to the natural scene environment

We are going to assume that the environment is made of random variables taken from a laplace (double exponential), gaussian, or uniform distribution. Gaussian and uniform numbers have non-positive kurtosis, so they will be the “noise” in the system, whereas the laplacian numbers will be the “structure”. We will then go through the cases for 1D, 2D, 2D with 2 eyes (or 4D).

1 1D

1.1 Laplace Distribution

Calculating the densities

$$f_x(x) = \frac{1}{2\lambda} e^{-|x|/\lambda} \quad (1.1)$$

$$y = x \cdot m \text{ (note: no sigmoid)} \quad (1.2)$$

$$z \equiv \sigma(y) \equiv \begin{cases} y & \text{if } y \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (1.3)$$

$$\text{(see section A.1) } f_y(y) = \frac{1}{2\lambda|m|} e^{-|y/m|/\lambda} \quad (1.4)$$

$$f_z(z) = \begin{cases} f_y(y) & \text{if } y > 0 \\ \frac{1}{2}\delta(y) & \text{if } y = 0 \\ 0 & \text{if } y < 0 \end{cases} \quad (1.5)$$

We will assume that m is positive, so we don't have to carry the absolute value through the calculations. Note that that any solution we find, say $m = m_o$, *must* be positive, because it represents $|m| = m_o$, but that the solution will really be $m = \pm m_o$.

Calculating the risk

$$R = \frac{1}{3}E[z^3] - \frac{1}{4}E^2[z^2] \quad (1.6)$$

$$= \frac{1}{3} \int_{-\infty}^{\infty} z^3 f_z(z) dz - \frac{1}{4} \left(\int_{-\infty}^{\infty} z^2 f_z(z) dz \right)^2 \quad (1.7)$$

$$E[z^2] = \int_{-\infty}^{\infty} z^2 f_z(z) dz \quad (1.8)$$

$$= \frac{1}{2\lambda m} \int_0^{\infty} z^2 e^{-z/m\lambda} dz \quad (1.9)$$

$$\text{(see section A.2) } = m^2 \lambda^2 \quad (1.10)$$

$$E[z^3] = \int_{-\infty}^{\infty} z^3 f_z(z) dz \quad (1.11)$$

$$= \frac{1}{2\lambda m} \int_0^{\infty} z^3 e^{-z/m\lambda} dz \quad (1.12)$$

$$= 3m^3 \lambda^3 \quad (1.13)$$

$$\boxed{R = m^3 \lambda^3 - \frac{1}{4} m^4 \lambda^4} \quad (1.14)$$

Fixed points

$$\frac{dR}{dm} = 3m^2\lambda^3 - m^3\lambda^4 \quad (1.15)$$

$$= m^2\lambda^3(3 - m\lambda) = 0 \quad (1.16)$$

$$\Rightarrow m = 0, \frac{3}{\lambda} \quad (1.17)$$

Stability

$$\frac{d^2R}{dm^2} = 6m\lambda^3 - 3m^2\lambda^4 \quad (1.18)$$

$$= 3m\lambda^3(2 - m\lambda) \quad (1.19)$$

$$= \begin{cases} \frac{d^2R}{dm^2}|_{m=0} = 0 & \text{unstable} \\ \frac{d^2R}{dm^2}|_{m=3/\lambda} < 0 & \text{stable} \end{cases} \quad (1.20)$$

So the only stable fixed point is $m = 3/\lambda$ and $\theta \equiv E[z^2] = 9$.

1.2 Gaussian Distribution

Calculating the densities

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} \quad (1.21)$$

$$y = x \cdot m \text{ (note: no sigmoid)} \quad (1.22)$$

$$z \equiv \sigma(y) \equiv \begin{cases} y & \text{if } y \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (1.23)$$

$$f_y(y) = \frac{1}{m\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2 m^2} \quad (1.24)$$

$$f_z(z) = \begin{cases} f_y(y) & \text{if } y > 0 \\ \frac{1}{2}\delta(y) & \text{if } y = 0 \\ 0 & \text{if } y < 0 \end{cases} \quad (1.25)$$

Calculating the risk

$$R = \frac{1}{3}E[z^3] - \frac{1}{4}E^2[z^2] \quad (1.26)$$

$$= \frac{1}{3} \int_{-\infty}^{\infty} z^3 f_z(z) dz - \frac{1}{4} \left(\int_{-\infty}^{\infty} z^2 f_z(z) dz \right)^2 \quad (1.27)$$

$$E[z^2] = \frac{1}{2}m^2\sigma^2 \quad (1.28)$$

$$E[z^3] = \sqrt{\frac{2}{\pi}}m^3\sigma^3 \quad (1.29)$$

$$\boxed{R = \frac{1}{3}\sqrt{\frac{2}{\pi}}m^3\sigma^3 - \frac{1}{16}m^4\sigma^4} \quad (1.30)$$

Fixed points

$$\frac{dR}{dm} = \sqrt{\frac{2}{\pi}}m^2\sigma^3 - \frac{1}{4}m^3\sigma^4 \quad (1.31)$$

$$= m^2 \sigma^3 \left(\sqrt{\frac{2}{\pi}} - \frac{1}{4} m \sigma \right) = 0 \quad (1.32)$$

$$\Rightarrow m = 0, \frac{4}{\sigma} \sqrt{\frac{2}{\pi}} \quad (1.33)$$

Stability

$$\frac{d^2 R}{dm^2} = 2\sqrt{\frac{2}{\pi}} m \sigma^3 - \frac{3}{4} m^2 \sigma^4 \quad (1.34)$$

$$= m \sigma^3 \left(2\sqrt{\frac{2}{\pi}} - \frac{3}{4} m \sigma \right) \quad (1.35)$$

$$= \begin{cases} \frac{d^2 R}{dm^2} \Big|_{m=0} = 0 & \text{unstable} \\ \frac{d^2 R}{dm^2} \Big|_{m=\frac{4}{\sigma} \sqrt{\frac{2}{\pi}}} < 0 & \text{stable} \end{cases} \quad (1.36)$$

So the only stable fixed point is $m = \frac{4}{\sigma} \sqrt{\frac{2}{\pi}}$ and $\theta = 16/\pi \sim 5.1$

1.3 Uniform Distribution

Calculating the densities

$$f_x(x) = \frac{1}{2a} \text{ in the range } [-a..a] \quad (1.37)$$

$$f_y(y) = \frac{1}{2ma} \text{ in the range } [-ma..ma] \quad (1.38)$$

$$f_z(z) = \begin{cases} \frac{1}{2} & \text{if } z = 0 \\ \frac{1}{2ma} & \text{in the range } [0..ma] \end{cases} \quad (1.39)$$

Calculating risk

$$E[z^2] = \int_0^{am} z^2 \frac{1}{2ma} \quad (1.40)$$

$$= \frac{a^2 m^2}{6} \quad (1.41)$$

$$E[z^3] = \frac{a^3 m^3}{8} \quad (1.42)$$

$$\boxed{R = \frac{a^3 m^3}{24} - \frac{a^4 m^4}{144}} \quad (1.43)$$

Fixed points

$$\frac{dR}{dm} = a^3 m^2 \left(\frac{1}{8} - \frac{am}{36} \right) \quad (1.44)$$

$$m = 0, \frac{9}{2a} \quad (1.45)$$

The stable fixed point is $m = 9/2a$ and $\theta = 27/8 \sim 3.4$.

2 2D

2.1 Laplace-Laplace Mixture in 2D

Calculating the densities. Remember, we are still assuming that all weights are positive, for the calculations.

$$y = \mathbf{x} \cdot \mathbf{m} = x_1 m_1 + x_2 m_2 \quad (2.46)$$

$$\equiv y_1 + y_2 \text{ (note: no sigmoid)} \quad (2.47)$$

$$z \equiv \sigma(y) \quad (2.48)$$

$$f_{x_i}(x_i) = \frac{1}{2\lambda} e^{-|x_i|/\lambda} \quad (2.49)$$

$$\text{(from section A.1)} \quad f_{y_i}(y_i) = \frac{1}{2\lambda m_i} e^{-|y_i|/m_i \lambda} \quad (2.50)$$

$$f_y(y) = \int_{-\infty}^{-\infty} f_{y_1}(y - y_2) f_{y_2}(y_2) dy_2 \quad (2.51)$$

$$= \frac{1}{4\lambda^2 m_1 m_2} \int_{-\infty}^{-\infty} e^{-\left|\frac{y-y_2}{m_1}\right|/\lambda} e^{-|y_2/m_2|/\lambda} dy_2 \quad (2.52)$$

$$\text{(from section A.2)} = \frac{1}{2\lambda(m_1 - m_2)(m_1 + m_2)} \left(m_1 e^{-|y|/m_1 \lambda} - m_2 e^{-|y|/m_2 \lambda} \right) \quad (2.53)$$

$$\text{Note: } f_y(y)|_{m_1=m_2} = \frac{1}{4\lambda m_2} e^{-|y|/m_2 \lambda} \left(\frac{|y|}{m_2 \lambda} + 1 \right) \quad (2.54)$$

Calculating the risk

$$R = \frac{1}{3} E[z^3] - \frac{1}{4} E^2[z^2] \quad (2.55)$$

$$= \frac{1}{3} \int_{-\infty}^{\infty} z^3 f_z(z) dz - \frac{1}{4} \left(\int_{-\infty}^{\infty} z^2 f_z(z) dz \right)^2 \quad (2.56)$$

$$E[z^2] = \frac{1}{2\lambda(m_1 - m_2)(m_1 + m_2)} (m_1 2\lambda^3 m_1^3 - m_2 2\lambda^3 m_2^3) \quad (2.57)$$

$$= \lambda^2 (m_1^2 + m_2^2) \quad (2.58)$$

$$E[z^3] = \frac{1}{2\lambda(m_1 - m_2)(m_1 + m_2)} (6m_1^5 \lambda^4 - 6m_2^5 \lambda^4) \quad (2.59)$$

$$= 3\lambda^3 \frac{m_1^5 - m_2^5}{m_1^2 - m_2^2} \quad (2.60)$$

$$R = \lambda^3 \frac{m_1^5 - m_2^5}{m_1^2 - m_2^2} - \frac{\lambda^4}{4} (m_1^2 + m_2^2)^2 \quad (2.61)$$

Fixed points

$$\frac{\partial R}{\partial m_1} = \frac{5\lambda^3 m_1^4}{m_1^2 - m_2^2} - \frac{2m_1 \lambda^3 (m_1^5 - m_2^5)}{(m_1^2 - m_2^2)^2} - \lambda^4 m_1 (m_1^2 + m_2^2) \quad (2.62)$$

$$= \lambda^3 m_1 \left(\frac{5m_1^3}{m_1^2 - m_2^2} - \frac{2(m_1^5 - m_2^5)}{(m_1^2 - m_2^2)^2} - \lambda (m_1^2 + m_2^2) \right) = 0 \quad (2.63)$$

$$\frac{\partial R}{\partial m_2} = \lambda^3 m_2 \left(\frac{5m_2^3}{m_1^2 - m_2^2} - \frac{2(m_1^5 - m_2^5)}{(m_1^2 - m_2^2)^2} - \lambda (m_1^2 + m_2^2) \right) = 0 \quad (2.64)$$

$$\mathbf{m} = \left(\begin{array}{c} 0 \\ 3/\lambda \end{array} \right), \left(\begin{array}{c} 3/\lambda \\ 0 \end{array} \right), \left(\begin{array}{c} \frac{5}{3\lambda} - \frac{1}{18\lambda} (15 \pm 15\sqrt{5}) \\ \frac{1}{18\lambda} (15 \pm 15\sqrt{5}) \end{array} \right) \quad (2.65)$$

The stable fixed points are

$$\mathbf{m} = \begin{pmatrix} 0 \\ 3/\lambda \end{pmatrix}, \begin{pmatrix} 3/\lambda \\ 0 \end{pmatrix} \quad (2.66)$$

$$\theta = 9 \quad (2.67)$$

2.2 Gaussian-Gaussian Mixture in 2D

Calculating the densities

$$f_y(y) = \int_{-\infty}^{\infty} f_{y_1}(y - y_2) f_{y_2}(y_2) dy_2 \quad (2.68)$$

$$= \frac{1}{m_1 m_2} \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{-(y-y_2)^2/2\sigma^2 m_1^2} e^{-y_2^2/2\sigma^2 m_2^2} \quad (2.69)$$

$$= \sqrt{\frac{1}{m_1^2 + m_2^2}} \sqrt{\frac{1}{2\pi\sigma^2}} e^{-y^2/2\sigma^2(m_1^2 + m_2^2)} \quad (2.70)$$

Calculating the risk

$$E[z^2] = \frac{\sigma^2}{2} (m_1^2 + m_2^2) \quad (2.71)$$

$$E[z^3] = \sqrt{\frac{2}{\pi}} \sigma^3 (m_1^2 + m_2^2)^{3/2} \quad (2.72)$$

$$R = \sqrt{\frac{2}{\pi}} \frac{\sigma^3}{2} (m_1^2 + m_2^2)^{3/2} - \frac{\sigma^4}{16} (m_1^2 + m_2^2)^2 \quad (2.73)$$

Fixed points

$$\frac{\partial R}{\partial m_1} = \sigma^3 m_1 \sqrt{\frac{2}{\pi}} (m_1^2 + m_2^2)^{1/2} - \frac{\sigma^4 m_1}{4} (m_1^2 + m_2^2) \quad (2.74)$$

$$\frac{\partial R}{\partial m_2} = \sigma^3 m_2 \sqrt{\frac{2}{\pi}} (m_1^2 + m_2^2)^{1/2} - \frac{\sigma^4 m_2}{4} (m_1^2 + m_2^2) \quad (2.75)$$

$$\mathbf{m} = \begin{pmatrix} \pm ir \\ r \end{pmatrix}, \begin{pmatrix} \left(\frac{\pi\sigma^2 r}{\pi}\right)^{1/2} \\ r \end{pmatrix}, \begin{pmatrix} 0 \\ \pm \frac{4}{\sigma} \sqrt{\frac{2}{\pi}} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.76)$$

There are no stable fixed points for this case. There is a series of fixed points, such that

$$|\mathbf{m}| = \frac{4}{\sigma} \sqrt{\frac{2}{\pi}} \quad (2.77)$$

$$\theta = 16/\pi \sim 5.1 \quad (2.78)$$

Once this length is achieved, all directions look the same, and have the same energy function.

2.3 Uniform-Uniform Mixture in 2D

Calculating the densities

$$f_y(y) = \int_{-\infty}^{\infty} f_{y_1}(y-y_2)f_{y_2}(y_2)dy_2 \quad (2.79)$$

$$= \frac{1}{2am_2} \int_{-am_2}^{am_2} f_{y_1}(y-y_2)dy_2 \quad (2.80)$$

Assume (for now) that $m_1 < m_2$ and $y > 0$. The distribution is symmetric, so I can symmetrize it later.

$$\text{for } y + am_1 < am_2: f_y(y) = \frac{1}{4a^2m_1m_2} y_2 \Big|_{-m_1a}^{m_1a} \quad (2.81)$$

$$= \frac{1}{2am_2} \quad (2.82)$$

$$\text{for } am_2 - am_1 < y < am_2 + am_1: f_y(y) = \frac{1}{2am_2} \int_{-am_1}^{am_2-y} \frac{1}{2am_1} \quad (2.83)$$

$$= \frac{m_1 + m_2 - y}{4am_1m_2} \quad (2.84)$$

This generalizes to the $m_1 > m_2$ and the $y < 0$ cases, giving

$$f_y(y) = \begin{cases} \frac{1}{2a \max(m_1, m_2)} & \text{if } |y| < a|m_2 - m_1| \\ \frac{m_1 + m_2 - |y|}{4am_1m_2} & \text{if } a|m_2 - m_1| < |y| < a(m_1 + m_2) \\ 0 & \text{otherwise} \end{cases} \quad (2.85)$$

Calculating the risk (assuming $m_2 > m_1$)

$$E[z^2] = \frac{a^2}{6} (m_1^2 + m_2^2) \quad (2.86)$$

$$E[z^3] = \frac{a^3}{40m_2} (5m_2^4 + 10m_1^2m_2^2 + m_1^4) \quad (2.87)$$

$$R = \frac{a^3}{720} (30m_2^4 + 60m_1^2m_2^2 + 6m_1^4 - 5am_2^5 - 10am_2^3m_1^2 - 5am_2m_1^4) \quad (2.88)$$

Fixed points

$$\frac{\partial R}{\partial m_1} = \frac{a^3m_1}{180m_2} (30m_2^2 + 6m_1^2 - 5am_2^3 + 5am_2m_1^2) = 0 \quad (2.89)$$

$$\frac{\partial R}{\partial m_2} = \frac{a^3}{360m_2^2} (45m_2^4 + 30m_1^2m_2^2 - 10am_2^5 - 10am_1^2m_2^3 - 3m_1^4) = 0 \quad (2.90)$$

$$\mathbf{m} = \left(\frac{0}{\frac{9}{2a}} \right), \left(\frac{\frac{1}{a}}{\pm 2\sqrt{5}} \right), \left(\frac{\frac{18}{5a}}{\pm 18} \right) \quad (2.91)$$

and the only stable fixed point is

$$\mathbf{m} = \left(\frac{\frac{18}{5a}}{\frac{5a}{18}} \right) \quad (2.92)$$

$$\theta = 324/75 \sim 4.3 \quad (2.93)$$

2.4 Laplace-Gaussian Mixture in 2D

Calculating the densities

$$f_y(y) = \frac{1}{m_1 m_2} \frac{1}{2\lambda\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-(y-y_2)^2/2\sigma^2 m_1^2} e^{-|y_2|/\lambda m_2^2} \quad (2.94)$$

$$= \frac{1}{m_1 m_2} \frac{1}{2\lambda\sqrt{2\pi\sigma^2}} \left(\int_{-\infty}^0 e^{-(y-y_2)^2/2\sigma^2 m_1^2} e^{y_2/\lambda m_2^2} + \int_0^{\infty} e^{-(y-y_2)^2/2\sigma^2 m_1^2} e^{-y_2/\lambda m_2^2} \right) \quad (2.95)$$

$$= \frac{1}{4\lambda m_2} e^{(-2\lambda m_2 y + \sigma^2 m_1^2)/2\lambda^2 m_2^2} \left(1 - \operatorname{erf} \left(\frac{1}{\sqrt{2}} \frac{-\lambda m_2 y + \sigma^2 m_1^2}{\sigma m_1 \lambda m_2} \right) \right) + \frac{1}{4\lambda m_2} e^{(2\lambda m_2 y + \sigma^2 m_1^2)/2\lambda^2 m_2^2} \left(1 - \operatorname{erf} \left(\frac{1}{\sqrt{2}} \frac{\lambda m_2 y + \sigma^2 m_1^2}{\sigma m_1 \lambda m_2} \right) \right) \quad (2.96)$$

Blech! Also, there seems to be some problem with this calculation, so I am not going to use it for now. This result motivated me to include the uniform distribution.

2.5 Laplace-Uniform Mixture in 2D

Calculating the densities

$$f_y(y) = \int_{-\infty}^{\infty} f_{y_1}(y - y_2) f_{y_2}(y_2) dy_2 \quad (2.97)$$

$$= \frac{1}{4\lambda a m_1 m_2} \int_{-a m_2}^{a m_2} e^{-|y-y_2|/m_1 \lambda} dy_2 \quad (2.98)$$

$$\equiv \frac{1}{4\lambda a m_1 m_2} \int_{-a m_2 - y}^{a m_2 - y} e^{-|u|/m_1 \lambda} du \quad (2.99)$$

assume, for now, that $y > 0$.

$$\text{if } y < a m_2: f_y(y) = \frac{1}{4\lambda a m_1 m_2} \left(\int_{-a m_2 - y}^0 e^{u/m_1 \lambda} du + \int_0^{a m_2 - y} e^{-u/m_1 \lambda} du \right) \quad (2.100)$$

$$= \frac{1}{4\lambda a m_1 m_2} \left(2\lambda m_1 - \lambda m_1 e^{(-a m_2 - y)/\lambda m_1} - \lambda m_1 e^{(-a m_2 + y)/\lambda m_1} \right) \quad (2.101)$$

$$\text{if } y > a m_2: f_y(y) = \frac{1}{4\lambda a m_1 m_2} \int_{-a m_2 - y}^{a m_2 - y} e^{u/m_1 \lambda} du \quad (2.102)$$

$$= \frac{1}{4\lambda a m_1 m_2} \left(\lambda m_1 e^{(a m_2 + y)/\lambda m_1} - \lambda m_1 e^{-(a m_2 + y)/\lambda m_1} \right) \quad (2.103)$$

which generalizes for the $y < 0$ case in the following

$$\text{if } |y| < a m_2: f_y(y) = \frac{1}{4a m_2} \left(2 - e^{(-a m_2 + |y|)/\lambda m_1} - e^{(-a m_2 - |y|)/\lambda m_1} \right) \quad (2.104)$$

$$\text{if } |y| > a m_2: f_y(y) = \frac{1}{4a m_2} \left(e^{(a m_2 + |y|)/\lambda m_1} - e^{-(a m_2 + |y|)/\lambda m_1} \right) \quad (2.105)$$

Calculating the risk

$$E[z^2] = \frac{1}{6} a^2 m_2^2 + \lambda^2 m_1^2 \quad (2.106)$$

$$E[z^3] = \frac{1}{8a m_2} \left(12\lambda^2 m_1^2 a^2 m_2^2 + a^4 m_2^4 + 24\lambda^4 m_1^4 (1 - e^{-a m_2/\lambda m_1}) \right) \quad (2.107)$$

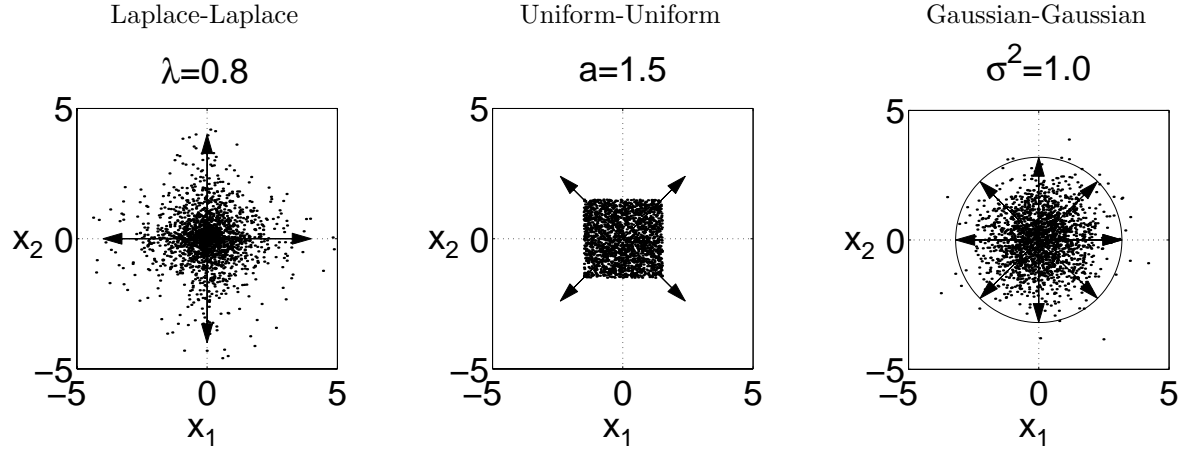
Expanded, the risk is

$$R = \frac{1}{2}\lambda^2 m_1^2 a m_2 + \frac{1}{24}a^3 m_2^3 + \lambda^4 m_1^4 \left(1 - e^{-am_2/\lambda m_1}\right) / a m_2 - \frac{1}{144}a^4 m_2^4 - \frac{1}{12}a^2 m_2^2 \lambda^2 m_1^2 - \frac{1}{4}\lambda^4 m_1^4 \quad (2.108)$$

$$R = \frac{1}{24am_2} \left(12\lambda^2 m_1^2 a^2 m_2^2 + a^4 m_2^4 + 24\lambda^4 m_1^4 \left(1 - e^{-am_2/\lambda m_1}\right)\right) - \frac{1}{4} \left(\frac{1}{6}a^2 m_2^2 + \lambda^2 m_1^2\right)^2 \quad (2.109)$$

3 Two eyes

The goal of these calculations is to somehow get a handle on some of the dynamics of the natural scene environment, as an alternative to the abstract input environment. It has turned out to be more complicated, even in the simple cases considered here, but we can still get some useful information. To start, however, let's look at pictures of the solutions for laplace-laplace, gaussian-gaussian, and uniform-uniform mixtures.



It is clear that BCM is finding the structure in the data in the form of directions of high kurtosis.

For two eyes we have the following

$$y = m_1 x_1^l + m_2 x_2^l + m_3 x_1^r + m_4 x_2^r \quad (3.110)$$

which reduces to the following, if the eyes see the same thing:

$$y = (m_1 + m_3)x_1 + (m_2 + m_4)x_2 \quad (3.111)$$

$$= m_1^{\text{eff}} x_1 + m_2^{\text{eff}} x_2 \quad (3.112)$$

We notice that we can use our solutions from the previous sections (Sections 2.1, 2.2, and 2.3) to find \mathbf{m}^{eff} , but that the solution is not uniquely defined. *Any* solution satisfying $\mathbf{m}^l + \mathbf{m}^r = \mathbf{m}^{\text{eff}}$ would work, so we have a continuum of solutions. Since the values of modification for each eye are *identical* (the output, y , is the same and the modification goes linearly with the input \mathbf{x}) the fixed point attained is determined by the *initial* difference in the two eye weights. Therefore, if we start the weights of small, we achieve the fixed point where $\mathbf{m}^l = \mathbf{m}^r = \mathbf{m}^{\text{eff}}/2$.

Normal Rearing, with two 2D eyes, can be modeled as a 2D Laplace-Laplace mixture (not 4D!) because the eyes see the same thing. This would yield the following fixed points:

$$\mathbf{m}^{\text{eff}} = \begin{pmatrix} \pm 3/2\lambda \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm 3/2\lambda \end{pmatrix} \quad (3.113)$$

$$\mathbf{m} = \begin{pmatrix} \pm 3/2\lambda \\ 0 \\ \pm 3/2\lambda \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm 3/2\lambda \\ 0 \\ \pm 3/2\lambda \end{pmatrix} \quad (3.114)$$

For Monocular Deprivation, the two eyes don't see the same thing, so we can't make this same simplification. We can, however, make use of the fixed points from normal rearing to do the job for us. Since, in the 4D case, two of the components of the fixed point weights are zero, the problem reduces to a two dimensional problem again: no modification of these weights will occur. Thus, we start from the following fixed points

$$\mathbf{m}_{\text{eff}} = \begin{pmatrix} \pm 3/2\lambda \\ \pm 3/2\lambda \end{pmatrix} \quad (3.115)$$

and evolve in an environment of structure (laplace distribution) in one input and noise (gaussian or uniform) in the other. Likewise, we can do the other deprivation protocols.

Experiment	Initial Conditions	Environment
Normal Rearing	$\mathbf{m}_{\text{eff}} = \begin{pmatrix} \epsilon \\ \epsilon \end{pmatrix}$	Laplace-Laplace
Monocular Deprivation	$\mathbf{m}_{\text{eff}} = \begin{pmatrix} 3/2\lambda \\ 3/2\lambda \end{pmatrix}$	Laplace-[Uniform/Gaussian]
Binocular Deprivation	$\mathbf{m}_{\text{eff}} = \begin{pmatrix} 3/2\lambda \\ 3/2\lambda \end{pmatrix}$	[Uniform-Uniform/Gaussian-Gaussian]
Reverse Suture	$\mathbf{m}_{\text{eff}} = \begin{pmatrix} 3/\lambda \\ \epsilon \end{pmatrix}$	[Uniform/Gaussian]-Laplace

3.1 Analysis of Monocular Deprivation (MD)

For MD, we start at the point

$$\mathbf{m} = \begin{pmatrix} 3/2\lambda \\ 0 \\ 3/2\lambda \\ 0 \end{pmatrix}$$

which gives us an effective 2D initial weight vector of

$$\mathbf{m}_{\text{eff}} = \begin{pmatrix} 3/2\lambda \\ 3/2\lambda \end{pmatrix}$$

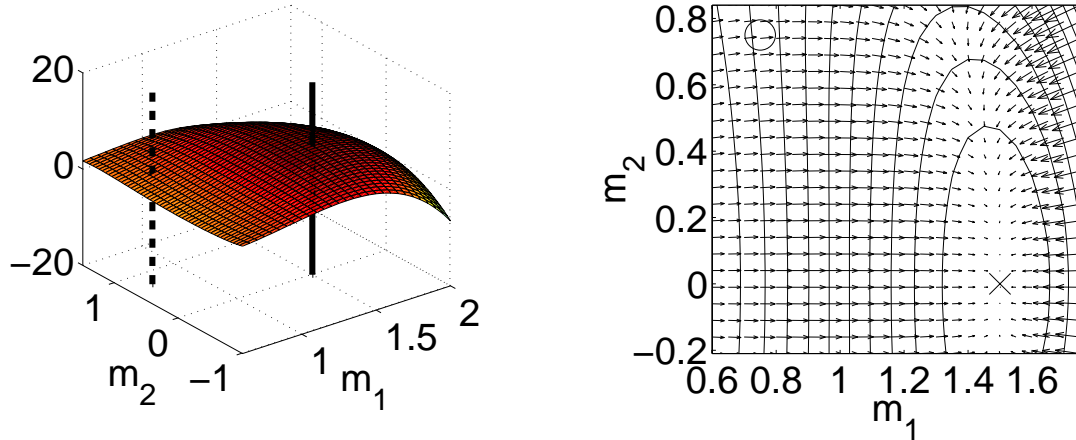
We are in the Laplace-Uniform¹ environment, with a risk function

$$R = \frac{1}{2}\lambda^2 m_1^2 a m_2 + \frac{1}{24} a^3 m_2^3 + \lambda^4 m_1^4 \left(1 - e^{-a m_2 / \lambda m_1}\right) / a m_2 - \frac{1}{144} a^4 m_2^4 - \frac{1}{12} a^2 m_2^2 \lambda^2 m_1^2 - \frac{1}{4} \lambda^4 m_1^4 \quad (3.116)$$

In order to proceed, it would help to look at the the energy function numerically, as well as the gradient, to see how the weights evolve. The following is the risk and gradient in Laplace-Uniform environment (monocular deprivation).

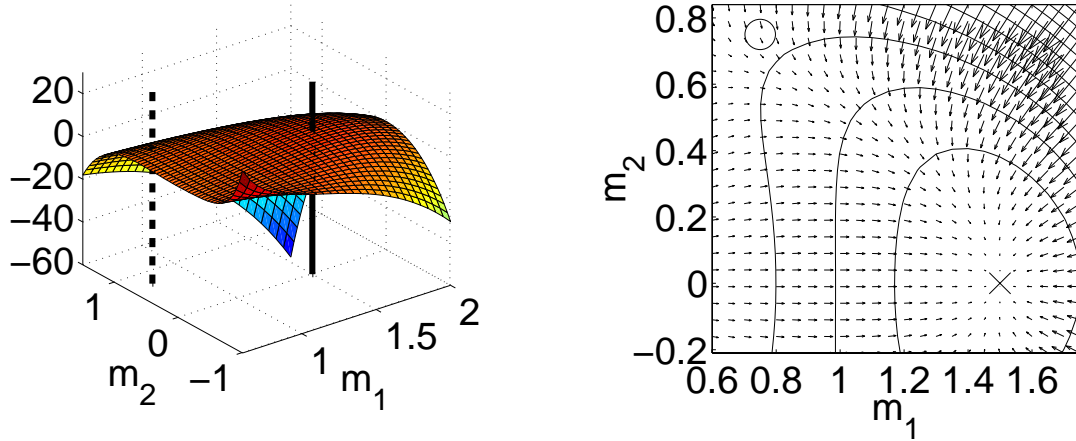
¹Or the Laplace-Gaussian environment, but we couldn't solve that one.

Risk for Laplace–Uniform ($\lambda=2.0, a=3.0$)



Weight 1 (m_1) is presented Laplace numbers and weight 2 (m_2) is presented uniform noise. The risk, shown on the left, is to be maximized. The dotted line shows the initial conditions, and solid line shows the top of the risk surface. The weight for the input receiving structure (m_1) grows to the 1D laplace solution, and the weight receiving noise (m_2) decays to zero. The gradient (left) shows the flow of the weights from the initial conditions (circle) to the fixed point (X). The graph shows that the weight modification is dominated by the m_1 approach to its fixed point value, and the decay of m_2 occurs only upon nearing this point. This is true for small noise. For larger noise, the dynamics are more complicated, as shown in the following.

Risk for Laplace–Uniform ($\lambda=2.0, a=6.0$)



We can now proceed to do the analysis using the *small noise* approximation.

If $a \ll \lambda$, (and we assume that the weights themselves are not particularly large), then we can expand the exponential in the risk function (Equation 3.116) to powers of $(a/\lambda)^2$.

$$R = \frac{1}{2}\lambda^2 m_1^2 a m_2 + \frac{1}{24}a^3 m_2^3 + \frac{\lambda^4 m_1^4}{a m_2} - \lambda^4 m_1^4 \left(1 - \frac{a m_2}{\lambda m_1} + \frac{a^2 m_2^2}{2\lambda^2 m_1^2}\right) / a m_2 - \frac{1}{144}a^4 m_2^4 - \frac{1}{12}a^2 m_2^2 \lambda^2 m_1^2 - \frac{1}{4}\lambda^4 m_1^4 \quad (3.117)$$

$$= \frac{1}{24}a^3 m_2^3 + \lambda^3 m_1^3 - \frac{1}{144}a^4 m_2^4 - \frac{1}{12}a^2 m_2^2 \lambda^2 m_1^2 - \frac{1}{4}\lambda^4 m_1^4 \quad (3.118)$$

The modification equations are then

$$\frac{d m_1}{d t} \equiv \frac{\partial R}{\partial m_1} = 3\lambda^3 m_1^2 - \frac{1}{6}a^2 \lambda^2 m_1 m_2^2 - \lambda^4 m_1^3 \quad (3.119)$$

$$\frac{dm_2}{dt} \equiv \frac{\partial R}{\partial m_2} = \frac{1}{8}a^3m_2^2 - \frac{1}{111}a^4m_2^3 - \frac{1}{6}a^2\lambda^2m_2m_1^2 \quad (3.120)$$

In the small noise approximation, the second term in Equation 3.119 would be small, and we'd end up with an equation identical to the weight modification in the one dimensional laplace environment (Equation 1.16). In other words, in small noise, the input with structure (the open eye) develops normally to the fixed point for the lower dimensional environment.

For the closed eye (m_2) the dynamics are quite different. If we keep only the last term in Equation 3.120, which is clearly larger than the others, and we assume that the other weight has converged quickly and is at the fixed point ($m_1 = 3/\lambda$), then we obtain a simple exponential decay of m_2 with $(1/6)a^2 = (\text{variance})/4$ as the exponent of decay. Keeping further terms yields qualitatively similar forms, but messier.

A Misc Math results

A.1 From Papoulis

$$y = g(x) \quad (A.121)$$

$$\text{(all possible solutions) } x_i \equiv g^{-1}(y) \quad (A.122)$$

$$f_y(y) = \frac{1}{|g'(x_1)|}f_x(x_1) + \frac{1}{|g'(x_2)|}f_x(x_2) + \dots \quad (A.123)$$

$$y = ax + b \quad (A.124)$$

$$f_y(y) = \frac{1}{|a|}f_x\left(\frac{y-b}{a}\right) \quad (A.125)$$

$$z = x + y \quad (A.126)$$

$$f_z(z) = \int_{-\infty}^{-\infty} f(z-y, y)dy \quad (A.127)$$

$$\text{if independent: } f(x, y) = f_x(x)f_y(y) \quad (A.128)$$

$$f_z(z) = \int_{-\infty}^{-\infty} f_x(z-y)f_y(y)dy \quad (A.129)$$

A.2 Some trivial integrals

Indefinite Exponential integrals

$$\begin{aligned} I(a) &\equiv \int_u^v e^{-x/a} dx = ae^{-x/a} \Big|_u^v \\ &= a \left(e^{-u/a} - e^{-v/a} \right) \end{aligned}$$

$$\int_u^v x e^{-x/a} dx = a \left(u e^{-u/a} - v e^{-v/a} \right) + a^2 \left(e^{-u/a} - e^{-v/a} \right)$$

$$\int_u^v x^2 e^{-x/a} dx = a \left(u^2 e^{-u/a} - v^2 e^{-v/a} \right) + 2a^2 \left(u e^{-u/a} - v e^{-v/a} \right) + 2a^3 \left(e^{-u/a} - e^{-v/a} \right)$$

$$\begin{aligned} \int_u^v x^3 e^{-x/a} dx &= a \left(u^3 e^{-u/a} - v^3 e^{-v/a} \right) + 3a^2 \left(u^2 e^{-u/a} - v^2 e^{-v/a} \right) + \\ &6a^3 \left(u e^{-u/a} - v e^{-v/a} \right) + 6a^4 \left(e^{-u/a} - e^{-v/a} \right) \end{aligned}$$

$$\int_u^v x^4 e^{-x/a} dx = a \left(u^4 e^{-u/a} - v^4 e^{-v/a} \right) + 4a^2 \left(u^3 e^{-u/a} - v^3 e^{-v/a} \right) + 12a^3 \left(u^2 e^{-u/a} - v^2 e^{-v/a} \right) + 24a^4 \left(u e^{-u/a} - v e^{-v/a} \right) + 24a^5 \left(e^{-u/a} - e^{-v/a} \right)$$

Definite Exponential integrals

$$\begin{aligned} \int_0^{\infty} e^{-x/a} dx &= a \\ \int_0^{\infty} x e^{-x/a} dx &= a^2 \\ \int_0^{\infty} x^2 e^{-x/a} dx &= 2a^3 \\ \int_0^{\infty} x^3 e^{-x/a} dx &= 6a^4 \\ \int_0^{\infty} x^4 e^{-x/a} dx &= 24a^5 \end{aligned}$$

Definite Gaussian integrals

$$\begin{aligned} \int_0^{\infty} e^{-(ax^2+bx+c)} dx &= e^{\frac{b^2}{4a}-c} \sqrt{\frac{\pi}{a}} \\ \int_0^{\infty} e^{-x^2/a^2} dx &= \frac{1}{2} a \pi^{1/2} \\ \int_0^{\infty} x e^{-x^2/a^2} dx &= \frac{1}{2} a^2 \\ \int_0^{\infty} x^2 e^{-x^2/a^2} dx &= \frac{1}{4} a^3 \pi^{1/2} \\ \int_0^{\infty} x^3 e^{-x^2/a^2} dx &= \frac{1}{2} a^4 \end{aligned}$$

B Some Code Segments

B.1 Maple

Integral for 2D Laplace (Equation 2.52)

```
assume(m1,positive);
assume(m2,positive);
assume(lambda,positive);

# y>0
simplify((1/(4*lambda^2*m1*m2))*
int(exp((y-y2)/(m1*lambda))*exp(-y2/(m2*lambda)),y2=y..infinity) +
int(exp(-(y-y2)/(m1*lambda))*exp(-y2/(m2*lambda)),y2=0..y) +
int(exp(-(y-y2)/(m1*lambda))*exp(y2/(m2*lambda)),y2=-infinity..0)));

# y<0
simplify((1/(4*lambda^2*m1*m2))*
int(exp((y-y2)/(m1*lambda))*exp(-y2/(m2*lambda)),y2=0..infinity) +
```

```
int(exp(-(y-y2)/(m1*lambda))*exp(y2/(m2*lambda)),y2=-infinity..y) +
int(exp((y-y2)/(m1*lambda))*exp(y2/(m2*lambda)),y2=y..0));
```

2D Laplace Fixed Points (Equation 2.65)

```
assume(lambda,positive);
assume(m1,positive);
assume(m2,positive);

s1:=diff(lambda^3*(m1^5-m2^5)/(m1^2-m2^2)-(lambda^4/4)*(m1^2+m2^2)^2,m1);
s2:=diff(lambda^3*(m1^5-m2^5)/(m1^2-m2^2)-(lambda^4/4)*(m1^2+m2^2)^2,m2);

solve({s1=0,s2=0},{m1,m2});
```

2D Gaussian Fixed Points (Equation 2.76)

```
assume(sigma,positive);
assume(m1,positive);
assume(m2,positive);

(1/(m1*m2))*(1/(2*Pi*sigma^2))*int(exp(-(y-y2)^2/(2*sigma^2*m1^2))*
exp(-y2^2/(2*sigma^2*m2^2)),y2=-infinity..infinity);

II:= y-> 1/(sqrt(2*Pi*sigma^2))*1/(sqrt(m1^2+m2^2))*exp(-y^2/(2*sigma^2*(m1^2+m2^2)));

z2:=simplify(int(z^2*II(z),z=0..infinity));
z3:=factor(simplify(int(z^3*II(z),z=0..infinity)));

R:=(1/3)*z3-(1/4)*z2^2;

s1:=simplify(diff(R,m1));
s2:=simplify(diff(R,m2));

solve({s1,s2},{m1,m2});

H:=hessian(R,[m1,m2]);
ev:=eigenvals(H); # must both be negative

simplify(subs({m1=0,m2=0},ev[1]));
simplify(subs({m1=0,m2=0},ev[2]));

simplify(subs({m1=4*sqrt(2)/(sqrt(Pi)*sigma),m2=0},ev[1]));
simplify(subs({m1=4*sqrt(2)/(sqrt(Pi)*sigma),m2=0},ev[2]));

simplify(subs({m2=4*sqrt(2)/(sqrt(Pi)*sigma),m1=0},ev[1]));
simplify(subs({m2=4*sqrt(2)/(sqrt(Pi)*sigma),m1=0},ev[2]));

simplify(subs({m1=0,m2=((Pi*sigma^2*m1-32)/Pi)^(1/2)/sigma},ev[1]));
simplify(subs({m1=4*sqrt(2)/(sqrt(Pi)*sigma),m2=0},ev[2]));
```

2D Uniform Fixed Points (Equation 2.91)

```
with(linalg);
```

```

assume(a,positive);
assume(m1,positive);
assume(m2,positive);

E2:=simplify(int(z^2*(1/(2*a*m2)),z=0..a*(m2-m1))+
  int(z^2*(a*(m1+m2)-z)/(4*a^2*m1*m2),z=a*(m2-m1)..a*(m2+m1)));
E3:=simplify(int(z^3*(1/(2*a*m2)),z=0..a*(m2-m1))+
  int(z^3*(a*(m1+m2)-z)/(4*a^2*m1*m2),z=a*(m2-m1)..a*(m2+m1)));

R:=(simplify((1/3)*E3-(1/4)*E2^2));
s1:=simplify(diff(R,m1));
s2:=simplify(diff(R,m2));

solve({s1,s2},{m1,m2});

H:=hessian(R,[m1,m2]);
ev:=eigenvals(H); # must both be negative

simplify(subs({m1=0,m2=9/(2*a)},ev[1]));
simplify(subs({m1=0,m2=9/(2*a)},ev[2]));

simplify(subs({m1=2*sqrt(5)/a,m2=2/a},ev[1]));
simplify(subs({m1=2*sqrt(5)/a,m2=2/a},ev[2]));

simplify(subs({m1=18/(5*a),m2=18/(5*a)},ev[1]));
simplify(subs({m1=18/(5*a),m2=18/(5*a)},ev[2]));

```

Integral for 2D Laplace-Gaussian Mixture (Equation 2.95)

```

assume(m1,positive);
assume(m2,positive);
assume(lambda,positive);

s1:=y->simplify((1/(2*lambda*m2))*(1/(m1*sqrt(2*Pi*sigma^2)))*
  (int(exp(-(y-y2)^2/(2*sigma^2*m1^2))*exp(-y2/(m2*lambda)),y2=0..infinity)));
s2:=y->simplify((1/(2*lambda*m2))*(1/(m1*sqrt(2*Pi*sigma^2)))*
  (int(exp(-(y-y2)^2/(2*sigma^2*m1^2))*exp(y2/(m2*lambda)),y2=-infinity..0)));

f:=y->simplify(s1(y)+s2(y));
f(y);

```

Integral for 2D Laplace-Uniform Mixture (Equation 2.100)

```

assume(m1,positive);
assume(m2,positive);
assume(lambda,positive);
assume(a,positive);

simplify((1/(2*a*m2))*(1/(2*lambda*m1))*(int(exp(-y/(m1*lambda)),y=0..a*m2)+
  int(exp(y/(m1*lambda)),y=-a*m2..0)));

f1:=y->factor((1/(4*lambda*m1*a*m2))*simplify(int(exp(u/(m1*lambda)),
  u=(-a*m2-y)..0)+int(exp(-u/(m1*lambda)),u=0..(a*m2-y))));
f2:=y->factor((1/(4*lambda*m1*a*m2))*simplify(int(exp(u/(m1*lambda)),
  u=(-a*m2-y)..(a*m2-y))));

```

```
# should be 1/2
simplify(int(f1(y),y=0..a*m2)+int(f2(y),y=a*m2..infinity));
```

B.2 MATLAB

Visualizing 1D Laplace (Equation 1.4)

```
lambda=1;
y=lambda*randlaplace([1 10000]);
[n,x]=hist(y,100); n=n/(x(2)-x(1))/sum(n);
f=1/(2*lambda)*exp(-abs(x)/lambda);
plot(x,n,'yo',x,f,'y-'); yr=yrange; yrange([0 yr(2)]);
```

Visualizing 1D Gaussian (Equation 1.24)

```
sigma=1;
y=sigma*randn([1 10000]);
[n,x]=hist(y,100); n=n/(x(2)-x(1))/sum(n);
f=1/(sqrt(2*pi*sigma^2))*exp(-x.^2/(2*sigma^2));
plot(x,n,'yo',x,f,'y-'); yr=yrange; yrange([0 yr(2)]);
```

Visualizing 1D Uniform (Equation 1.38)

```
a=2;
y=2*a*(rand([1 10000])-.5);
[n,x]=hist(y,100); n=n/(x(2)-x(1))/sum(n);
f=(1/(2*a))*(abs(x)<=a);
plot(x,n,'yo',x,f,'y-'); yr=yrange; yrange([0 yr(2)]);
```

Visualizing 2D Mixture of Laplace-Laplace (Equation 2.53)

```
m1=1; m2=2; lambda=1;
y=[m1 m2]*lambda*randlaplace([2 10000]);
[n,x]=hist(y,100); n=n/(x(2)-x(1))/sum(n);

f=1/(2*lambda*(m1-m2)*(m1+m2))*(m1*exp(-abs(x)/(m1*lambda))-m2*exp(-abs(x)/(m2*lambda)));

plot(x,n,'yo',x,f,'y-'); yr=yrange; yrange([0 yr(2)]);
```

Visualizing 2D Mixture of Gaussian-Gaussian (Equation 2.70)

```
m1=1; m2=2; sigma=2;
y=[m1*sigma m2*sigma]*randn([2 10000]);
[n,x]=hist(y,100); n=n/(x(2)-x(1))/sum(n);

f=1/sqrt(2*pi*sigma^2)*1/sqrt(m2^2+m1^2)*exp(-x.^2/(2*sigma^2*(m1^2+m2^2)));

plot(x,n,'yo',x,f,'y-'); yr=yrange; yrange([0 yr(2)]);
```

Visualizing 2D Mixture of Uniform-Uniform (Equation 2.85)

```
a=1; m1=1; m2=2;
y=2*a*[m1 m2]*(rand([2 10000])-.5);
```

```

[n,x]=hist(y,100); n=n/(x(2)-x(1))/sum(n);

f= (1/(2*a*max(m1,m2)))*(abs(x)<a*abs(m2-m1))+ ...
    ((m2-abs(x)+m1)/(4*a*m1*m2)).* ( (abs(x)>a*abs(m2-m1)) & (abs(x)<a*(m2+m1)) );

plot(x,n,'o',x,f,'y-'); yr=yrange; yrange([0 yr(2)]);

```

Visualizing 2D Mixture of Laplace-Uniform (Equation 2.104)

```

a=5; m1=1; m2=2; lambda=1;
y=sum([m2*a*2*(rand([1 10000])-.5);m1*lambda*randlaplace([1 10000]) ]);

[n,x]=hist(y,100); dx=(x(2)-x(1)); n=n/(x(2)-x(1))/sum(n);

f1=(1/(4*a*m2))*(2-exp( -(a*m2+abs(x))/(m1*lambda) )- ...
    exp( -(a*m2-abs(x))/(m1*lambda) ) ).*(abs(x)<(a*m2));
f2=(1/(4*a*m2))*(exp( (a*m2-abs(x))/(m1*lambda) )- ...
    exp( -(a*m2+abs(x))/(m1*lambda) ) ).*(abs(x)>=(a*m2));
f=f1+f2;

c1=(1/(4*a*m2))*exp( (a*m2)/(m1*lambda) );
c2=(1/(4*a*m2))*exp( -(a*m2)/(m1*lambda) );
c3=(2/(4*a*m2));

f1=(c3-c2*(exp(-abs(x)/(m1*lambda))+exp(abs(x)/(m1*lambda)))).*(abs(x)<(a*m2));
f2=(c1-c2)*exp( -abs(x)/(m1*lambda) ).*(abs(x)>=(a*m2));

f=f1+f2;

plot(x,n,'yo',x,f,'y-'); yr=yrange; yrange([0 yr(2)]);

```