Brian Blais' Homemade Guide to the Harmonic Oscillator

This is an informal guide to understanding oscillatory systems. It takes a very practical approach, using recipes and examples, to give the reader some facility in solving the equations for oscillatory systems. Hopefully, once the reader has worked these problems out for herself, the full understanding of the topic will shortly follow.

## 1 Why the Harmonic Oscillator?

Everyone who initially studies the harmonic oscillator is immediately barraged with examples of pendula and blocks with springs. If the only significance of the harmonic oscillator lay in these examples it would be pretty limiting. In physics, the harmonic oscillator is an extremely important problem, for a simple reason.

Say, for example, you have a potential, $V(\boldsymbol{x})$, for some system and you want to find the equations governing the motion of a particle under the influence of this potential. You might have a very difficult time solving this problem, especially if the potential is complicated. You then might think to look at the behavior of the particle around the minimum of this energy, which would be the behavior near equilibrium. One can do this simply by performing a Taylor expansion of the potential, $V$, about a minimum $\boldsymbol{x}_o$. For clarity we look only at one dimension.

$$\begin{aligned} V(x) &= V(x_o) + (x - x_o)\left.\frac{dV(x)}{dx}\right|_{x=x_o} + (x-x_o)^2 \frac{1}{2!}\left.\frac{d^2V(x)}{dx^2}\right|_{x=x_o} + \cdots \\ &\approx V(x_o) + (x - x_o)\left.\frac{dV(x)}{dx}\right|_{x=x_o} + (x-x_o)^2 \frac{1}{2!}\left.\frac{d^2V(x)}{dx^2}\right|_{x=x_o} \end{aligned}$$

but at a minimum $\left.\frac{dV(x)}{dx}\right|_{x=x_o} = 0$, so we have

$$V(x) \approx V(x_o) + (x-x_o)^2 \frac{1}{2!}\left.\frac{d^2V(x)}{dx^2}\right|_{x=x_o}$$

which is exactly the harmonic oscillator potential.[1]

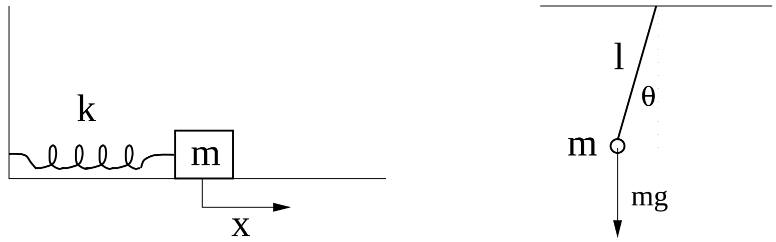
Essentially what we have shown is that (nearly) *any* system close to equilibrium can be approximated as a harmonic oscillator. This makes the problem a fundamental one in all fields of physics. Luckily, the harmonic oscillator is a problem which can be solved exactly (perhaps the only one in physics), so physicists take delight in mapping every problem to the harmonic oscillator.

## 2 Two Simple Examples

In this section we demonstrate the equivalence between the two simplest harmonic oscillator systems: the pendulum, and the block on a spring. All the examples afterwards are block-spring examples. What we want to do is to be able to predict the positions and velocities of the objects in our system as a function of time. Along the way, by studying simple examples, we will be able to understand in a fundamental way the behavior of much larger systems.

The two systems we want to look at are pictured below

[1] One can see this by remembering that the force is the negative gradient of the potential, which would yield $F(x) \approx -(\text{constant})(x - x_o)$




We will deal with each on in turn.

## 2.1 Block and Spring

- write down $\boldsymbol{F} = m\boldsymbol{a} = m\ddot{\boldsymbol{x}}$

It is assumed that the restoring force for a spring is $F = -kx$. When the block, pictured above, is pulled away from the wall there is a force in the negative $x$ direction. When is it pushed towards the wall there is a force in the positive $x$ direction. The positive direction of $x$ is depicted by the direction of the arrow shown above, and the $x = 0$ point is taken at the equilibrium point (where the force is zero). This will be the convention in all of the examples.

The differential equation we need to solve now is simply

$$m\ddot{x} = -kx \tag{2.1}$$

- guess a form for $\boldsymbol{x}(t)$ and plug into this differential equation

We know that the block-spring system oscillates, so it is prudent to try something like $x(t) = A\cos(\omega t), x(t) = B\sin(\omega t), x(t) = C\exp(i\omega t), ...$. Each of these functions is oscillatory, and would serve as solutions equally well as long as we satisfy the initial conditions. In general, the solution would be a sum of these oscillatory functions, but we will deal with that later.

Let's guess: $x(t) = A\cos(\omega t)$ and plug into Equation 2.1.

$$\begin{aligned} x(t) &\equiv A\cos(\omega t) \\ m\frac{d^2x}{dt^2} &= -kx \\ -m\omega^2(A\cos(\omega t)) &= -k(A\cos(\omega t)) \end{aligned} \tag{2.2}$$

- solve for the constants in $\boldsymbol{x}(t)$ in terms of the known quantities: masses, spring constants, initial conditions ($\boldsymbol{x}(0)$ and $\dot{\boldsymbol{x}}(0)$), etc...

From the Equation 2.2 one can clearly see that $\omega = \sqrt{k/m}$. $A$, is determined from the initial conditions. If we start the block at $x = 4$ at $t = 0$ then $x(0) = 4 = A\cos(0) = A$. Sometimes we will start the system already moving, so we would have to specify $\dot{x}(0)$ also, but we will not deal with this here.

- identify the physical significance of the constants

The guess of the solution demands that $\omega$ is the frequency of the oscillations. One can see that the value of this frequency is determined completely by the constants of the system: the mass and spring constant.

Because $A = x(0)$ (if the initial velocity,$\dot{x}(0)$, is zero), we can interpret $A$ as the amplitude of oscillations. It is interesting to note that it does not depend on the physical characteristics of the system, but only on the initial position of the block.

- make sure that the solution matches one's intuition

The solution we finally arrived at, for the block-spring problem, is

$$x(t) = A\cos(\omega t) = x(0)\cos\left(\sqrt{\frac{k}{m}}t\right) \tag{2.3}$$

Does this make sense? There are two easy ways of checking an answer, and one should get in the habit of using these ways when one has answered *any* problems. It takes very little time to do these checks, and they can quickly save you a lot of headaches.

- check units

  This is a very simple check, and will often tell one the physical meaning of particular quantities. In Equation 2.3 we note the following things:

  · $x$ has units of distance, $t$ has units of time

  · functions like $\cos(\cdot), \sin(\cdot), \log(\cdot)$, and $\exp(\cdot)$ return *dimensionless* values

  · parameters in such functions *must be dimensionless*

  From these basic statements, we have the following conclusions:

  · $A$ must have the same units as $x$, which is distance. This makes sense, because $A$ is a position amplitude.

  · $\omega t$ must be dimensionless, which implies that $\omega$ has units of 1/(time). This makes sense, because $\omega$ is a frequency.

  · from $\omega^2 = (k/m)$ it follows that $(k/m)$ must have units of $1/(\text{time})^2$.

  · from $m\ddot{x} = -kx$ it also follows that $(k/m)$ must have units of $1/(\text{time})^2$.

  This verifies the units of all unknown quantities directly from the initial equations we started with.

- check limiting cases

  Limiting cases are simplified cases of the solutions which are easy to interpret physically. They often include plugging in very small or very large values for parameters in the solution. In our case we can ask ourselves what happens when the mass gets very large, or very small, or what happens when the spring constant is large, or small.

  · mass is large: $m \gg 1$. $\omega$, the frequency of oscillations, becomes quite small.

  · mass is small: $m \ll 1$. $\omega$, the frequency of oscillations, becomes quite large.

  · spring is very stiff: $k \gg 1$. $\omega$, the frequency of oscillations, becomes quite large (vibrations are fast).

  · spring is very loose: $k \ll 1$. $\omega$, the frequency of oscillations, becomes quite small (vibrations are slow).

  These cases are what we'd expect given what we know about masses and springs.

We can now say that the units of the solution are correct, and it has the correct types of limiting behavior. We can also say that we understand this simple system quite well.

### 2.2 Pendulum

- write down $\boldsymbol{F} = m\boldsymbol{a} = m\ddot{\boldsymbol{x}}$

In this case $F = ma$ becomes the angular equivalent $\boldsymbol{\tau} = \boldsymbol{I\alpha}$, where the moment of inertia $I$ for a point mass a distance $l$ away is $ml^2$ and the torque $\boldsymbol{\tau}$ is $\boldsymbol{F} \cdot \boldsymbol{l}$. This yields

$$(mg\sin\theta)l = ml^2\ddot{\theta}$$

and with the small angle approximation, $\theta \ll 1 \Rightarrow \sin\theta \approx \theta$, we have

$$\begin{aligned} \ddot{\theta} &= -\frac{g}{l}\theta \text{ or with } x = l\sin\theta \approx l\theta \\ \ddot{x} &= -\frac{g}{l}x \end{aligned} \tag{2.4}$$

We note that Equation 2.4 is *identical in form* to the spring differential equation, Equation 2.1. This allows is to use the *same* solution, using $\omega^2 = g/l$ instead of $\omega^2 = k/m$. The only thing we need to do now is to check units and see if the solution itself makes sense in the limiting cases, both of which are simple to do. Though we don't do it here, it is *strongly* urged that the reader do it for all problems. A quick check can save hours of redoing algebra.

### 2.3 Summary of Two Examples

In this section we have seen the equivalence of two common oscillatory problems, the block-spring problem and the pendulum. We have outlined the general procedure for solving these simple problems.

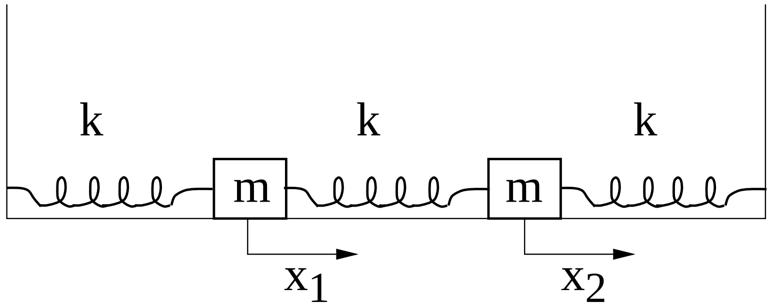
- write down $\boldsymbol{F} = m\ddot{\boldsymbol{x}}$ for the particular system
- guess a form for $\boldsymbol{x}(t)$ and plug into this differential equation
- solve for the constants in $\boldsymbol{x}(t)$ in terms of the known quantities: masses, spring constants, initial conditions ($\boldsymbol{x}(0)$ and $\dot{\boldsymbol{x}}(0)$), etc...
- identify the physical significance of the constants
- make sure that the solution matches one's intuition

As one does more of these problems, it becomes *much* easier guessing solutions, but one finds fairly quickly that more complicated solutions require a little bit more effort. We continue now with a more complicated situation.

## 3 A Coupled Example

In this section we outline the procedure for dealing with coupled oscillations. Though we will use a lot of linear algebra, it is not the intention of this guide to be either an introduction to linear algebra, nor as a reference to the general properties of matrices. Instead it gives a recipe that can be followed when solving oscillatory systems. Definitions of terms (such as *determinant*) can be found in any introductory linear algebra text or in a mechanics textbook.

The system we'd like to solve is pictured below.




What we are after are the functions $x_1(t)$ and $x_2(t)$ which govern the motion of the two blocks coupled together with springs. We will assume that the initial velocities of the blocks are zero, or $\dot{x_1}(0) = \dot{x_2}(0) = 0$, and that we know the values of the masses, both of which are $m$, the spring constants, both of which are $k$ in this case, and the initial positions of blocks 1 and 2, namely $x_1(0)$ and $x_2(0)$.

The recipe for solving this problem is merely to get the equations describing the problem to look like those in Section 2 so we can quickly use the solutions found earlier. This involves a coordinate transformation, because the original coordinates, $x_1$ and $x_2$, are coupled together. We need coordinates which behave like individual, distinct, uncoupled harmonic oscillators. These new coordinates will be called the *normal mode coordinates*, and their physical meaning will be explained later.

There are two straightforward ways of obtaining the coordinate transformation, so we will go through both ways in detail to show how they are related. The first method is the quickest way to obtain the answer, but it is not always possible. The second method involves more notation and algebra, but is a step-by-step method which will work for any size system, and is thus more generally applicable.

### 3.1 Method 1: Using a Clever Substitution

- write down the differential equations $F_i(x_1, x2, ...) = m_i\ddot{x}_i$ for all blocks $i$

For the system pictured above we can immediately write down the equations for the force on the two blocks. See Appendix A.1 for hints on getting the signs correct.

$$
\begin{aligned}
F_1 &= m\ddot{x}_1 = -kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2 \\
F_2 &= m\ddot{x}_2 = -kx_2 - k(x_2 - x_1) = kx_1 - 2x_2
\end{aligned}
$$

If we do as we did before, plug in a simple $x_1(t) = A_1\cos(\omega t)$ and $x_2(t) = A_2\cos(\omega t)$ we would get a quadratic equation for $\omega$ which would have two solutions (the reader should verify this). This would imply that somehow 2 frequencies are needed to describe the system, and that the behavior of $x_1$ and $x_2$ involves a mixture of the 2 frequencies. The next step involves unmixing the behavior so we can deal with each frequency individually.

- make a clever substitution $q_j = f(x_1, x2, ...)$ to replace all $x_j$ and obtain new differential equations $m_i q_i = -\omega_i^2 q_i$

The clever substitution we are going to make is

$$
\begin{aligned}
q_1 &\equiv x_1 + x_2 \\
q_2 &\equiv x_1 - x_2
\end{aligned}
$$

which immediately yields

$$\begin{aligned} m\ddot{q}_1 &= m(\ddot{x}_1 + \ddot{x}_2) = (-2k + k)x_1 + (k - 2k)x_2 = -kq_1 \\ m\ddot{q}_2 &= m(\ddot{x}_1 - \ddot{x}_2) = (-2k - k)x_1 + (k + 2k)x_2 = -3kq_2 \end{aligned}$$

Each of these equations is identical to the simple harmonic oscillator differential equation (Equation 2.1).

- solve for $q_i(t)$ for all $i$. using the solutions earlier, $q_i(t) = Q_i \cos(\omega_i t)$,each $q_i$ oscillating with a single frequency $\omega_i$. Use the initial conditions to obtain the unknown coefficients.

We can now say that the behavior of $q_i$ are

$$\begin{aligned} q_1 &= Q_1 \cos(\omega_1 t) = Q_1 \cos\left(\sqrt{\frac{k}{m}}t\right) \\ q_2 &= Q_2 \cos(\omega_2 t) = Q_2 \cos\left(\sqrt{\frac{3k}{m}}t\right) \end{aligned}$$

From the initial conditions one directly finds $Q_1 = (x_1(0) + x_2(0))$ and $Q_2 = (x_1(0) - x_2(0))$.

- use the clever substitution to restore the equations to ones involving $x_i$.

From the substitution earlier, we find $x_1 = (q_1 + q_2)/2$ and $x_2 = (q_1 - q_2)/2$ which yields the final solution

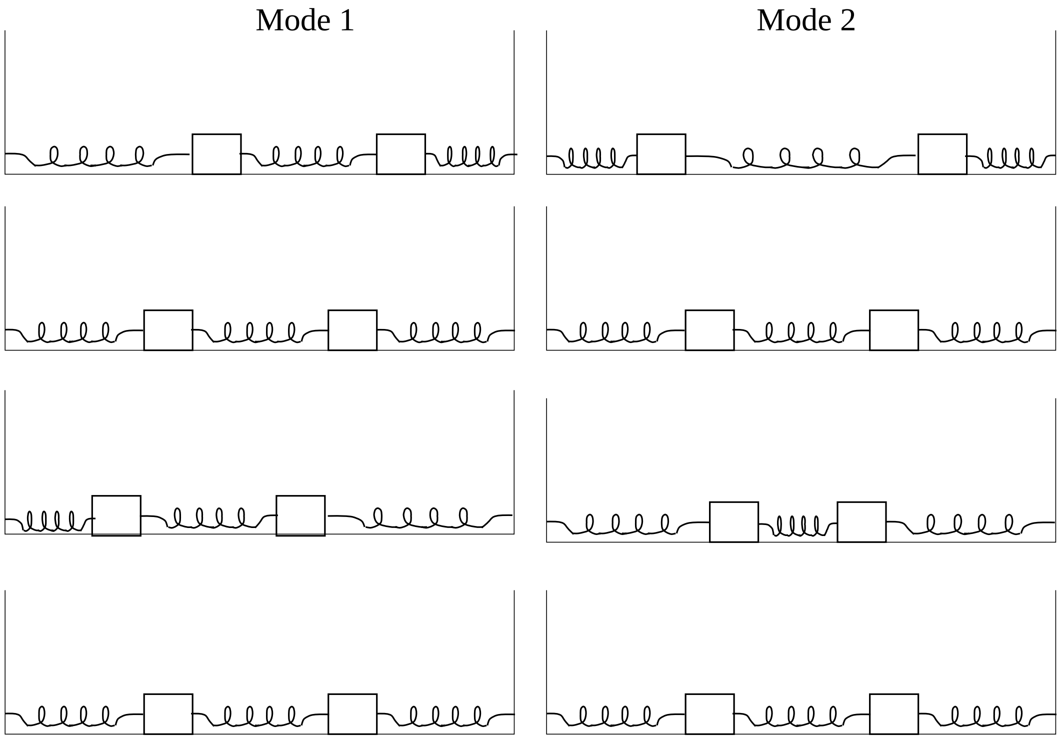
$$\begin{aligned} x_1(t) &= \frac{x_1(0) + x_2(0)}{2} \cos\left(\sqrt{\frac{k}{m}}t\right) + \frac{x_1(0) - x_2(0)}{2} \cos\left(\sqrt{\frac{3k}{m}}t\right) \\ x_2(t) &= \frac{x_1(0) + x_2(0)}{2} \cos\left(\sqrt{\frac{k}{m}}t\right) - \frac{x_1(0) - x_2(0)}{2} \cos\left(\sqrt{\frac{3k}{m}}t\right) \end{aligned} \quad (3.5)$$

- identify the physical significance of the solution and make sure that it matches one's intuition

Equation 3.5 is quite complicated, so it might be difficult to understand exactly what it is telling us. It is quite easy to plug in $t = 0$ and see that the initial conditions are satisfied, as a quick check. If they weren't, then we would have to review the algebra. In order to truly understand the solution we will explore the meaning of the substitution variables $q_i$.

The significance of the variables $q_i$ can be seen in the following way. If $q_1(0) = Q_1 = 0$ then the only variable which is oscillating is $q_2$, and it is oscillating with the particular frequency $\sqrt{3k/m}$. This implies that somehow the entire system is oscillating at that frequency, or in other words, some combination of $x_1$ and $x_2$ is oscillating with a particular frequency, not a mixed one. Likewise, if $q_2(0) = Q_2 = 0$ then a different combination of $x_1$ and $x_2$ is oscillating with a particular frequency. What is the meaning of these combinations? Each combination of the motions in the system such that the system oscillates with a single frequency is called a *normal mode*, and the frequency associated with it the *normal mode frequency.*

In the system we are looking at, there are two normal modes with frequencies $\sqrt{k/m}$ and $\sqrt{3k/m}$ respectively. The first one occurs when $q_2(0) = 0$ or $x_1(0) - x_2(0) = 0$ or when the blocks start off with equal displacements. The other occurs when $q_1(0) = 0$ or $x_1(0) + x_2(0) = 0$ or when the blocks start off with opposite displacements. The two modes are pictured below. It should be clear that the mode where the initial displacement is the same for both blocks has a lower frequency than the other one. We are now ready to discuss the second method.




## 3.2 Method 2: The Eigenvalue Problem

As before,

- write down the differential equations $F_i(x_1, x2, ...) = m_i\ddot{x}_i$ for all blocks $i$

$$\begin{aligned} F_1 &= m\ddot{x}_1 = -kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2 \\ F_2 &= m\ddot{x}_2 = -kx_2 - k(x_2 - x_1) = kx_1 - 2x_2 \end{aligned}$$

- place the equations in matrix notation: $\boldsymbol{M}\ddot{\boldsymbol{x}} = \boldsymbol{\Omega}\boldsymbol{x}$

Placing the equation is matrix notation is straightforward, and will allow us to deal with multiple equations in a unified manner. The vector $\boldsymbol{x}$ is the vector of positions, the matrix $\boldsymbol{M}$ is the mass matrix, and the matrix $\boldsymbol{\Omega}$ is the interaction matrix. Now we can make the formal definitions.

$$\begin{aligned} \text{(vector of positions)} &\equiv \boldsymbol{x} \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \text{(vector of accelerations)} &\equiv \ddot{\boldsymbol{x}} \equiv \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} \\ \text{(mass matrix)} &\equiv \boldsymbol{M} \equiv \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \end{aligned}$$

$$\text{(interaction matrix)} \quad \equiv \quad \boldsymbol{\Omega} = \begin{pmatrix} -2k & k \\ k & -2k \end{pmatrix}$$

so the matrix equation looks like

$$\begin{aligned} \boldsymbol{M}\ddot{\boldsymbol{x}} &= \boldsymbol{\Omega}\boldsymbol{x} \\ \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} &= \begin{pmatrix} -2k & k \\ k & -2k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

- solve the matrix equation $[\boldsymbol{\Omega} - \boldsymbol{M}(-\omega^2)]\boldsymbol{q} = 0$ for all possible values $\omega$ and all possible vectors $\boldsymbol{q}$ which satisfy the equation.

The equation $[\boldsymbol{\Omega} - \boldsymbol{M}(-\omega^2)]\boldsymbol{q} = 0$ can be seen as a matrix correlate to Equation 2.2. For some vector(s) $\boldsymbol{q}$, the system behaves as a single harmonic oscillator with frequency $\omega$. Each solution vector $\boldsymbol{q}$ will give us a normal mode, a coordinate combination which oscillates with one frequency. How do we solve this matrix equation? This is done in the following two steps.

· solving $\det|\boldsymbol{\Omega} - \boldsymbol{M}(-\omega^2)| = 0$, which gives the possible values for $\omega$

$$\begin{aligned} \det\left|\begin{pmatrix} -2k & k \\ k & -2k \end{pmatrix} - \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix}\right| &= 0 \\ &= \det\left|\begin{pmatrix} -2k + m\omega^2 & k \\ k & -2k + m\omega^2 \end{pmatrix}\right| \\ &= (-2k + m\omega^2)(-2k + m\omega^2) - k^2 \\ &= 3k^2 - 4km\omega^2 + m^2\omega^4 \\ (m\omega^2 - k)(m\omega^2 - 3k) &= 0 \end{aligned}$$

which yields the two values for $\omega$ which we found with Method 1.

$$\begin{aligned} \omega_1 &= \sqrt{\frac{k}{m}} \\ \omega_2 &= \sqrt{\frac{3k}{m}} \end{aligned}$$

· solve the matrix equation $[\boldsymbol{\Omega} - \boldsymbol{M}(-\omega^2)]\boldsymbol{q} = 0$ component-wise, to obtain a valid vector $\boldsymbol{q}$ for each valid $\omega$ value

We need to find the valid vectors $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$, which are associated with the frequencies $\omega_1$ and $\omega_2$. To do this we plug in a vector $(a_1 \;\; a_2)$ into the matrix equation for each $\omega$, and find the relation between the components. We will also demand that this vector be *normalized* $(a_1^2 + a_2^2 = 1)$, for reasons which will become clear as we proceed.

*Solving for* $\boldsymbol{q}_1$:

$$\begin{aligned} \boldsymbol{\Omega}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} &= \boldsymbol{M}(-\omega_1^2)\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\ \begin{pmatrix} -2k & k \\ k & -2k \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} &= -\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\left(\frac{k}{m}\right)\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \end{aligned}$$

which gives the following pair of equations specifying the components of $(a_1 \;\; a_2)$.

$$\begin{aligned} -2ka_1 + ka_2 &= \frac{-mka_1}{m} \Rightarrow a_1 = a_2 \\ a_1 - 2ka_2 &= \frac{-mka_2}{m} \Rightarrow a_1 = a_2 \end{aligned}$$

We notice that there aren't enough equations to specify $a_1$ and $a_2$ uniquely, so we add the normalization condition.

$$a_1^2 + a_2^2 = 1 \Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We have now found

$$\boldsymbol{q}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

*Solving for* $\boldsymbol{q}_2$:

$$\begin{aligned} \boldsymbol{\Omega} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} &= \boldsymbol{M}(-\omega_2^2) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\ \begin{pmatrix} -2k & k \\ k & -2k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} &= -\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \left(\frac{3k}{m}\right) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \end{aligned}$$

which gives the following equations specifying the components of $(a_1 \;\; a_2)$.

$$\begin{aligned} -2ka_1 + ka_2 &= \frac{-m3ka_1}{m} \Rightarrow a_1 = -a_2 \\ a_1 - 2ka_2 &= \frac{-m3ka_2}{m} \Rightarrow a_1 = -a_2 \\ \text{(normalization) } a_1^2 + a_2^2 &= 1 \Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$

We have now found

$$\boldsymbol{q}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

To denote the fact that they are normalized (ie they are unit vectors), we will refer to these two vectors as $\hat{\boldsymbol{q}}_1$ and $\hat{\boldsymbol{q}}_2$. They are also called *normal modes*, and *eigenvectors*.

- use superposition to restore the equations to ones involving $x_i$

This involves a few steps.

  · write down the solution vector $\boldsymbol{x}(t)$ in terms of the vectors $\hat{\boldsymbol{q}}_i$

The vectors $\hat{\boldsymbol{q}}_1$ and $\hat{\boldsymbol{q}}_2$ are *orthogonal*, which lets us express any other vectors in terms of these two vectors. The final solutions, $x_1(t)$ and $x_2(t)$ can be seen as a vector $\boldsymbol{x}(t)$, which can then be expressed as a superposition (ie. sum) of the vectors $\hat{\boldsymbol{q}}_i$. In vector notation we can write it like

$$\begin{aligned}
\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} &= x_1(t)\hat{\boldsymbol{x}}_1 + x_2(t)\hat{\boldsymbol{x}}_2 = x_1(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= q_1(t)\hat{\boldsymbol{q}}_1 + q_2(t)\hat{\boldsymbol{q}}_2 \\
&= Q_1\cos(\omega_1 t)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + Q_2\cos(\omega_2 t)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= Q_1\cos(\sqrt{k/m}t)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + Q_2\cos(\sqrt{3k/m}t)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}
\end{aligned}$$

where we have introduced the unit vectors $\hat{\boldsymbol{x}}_1$ and $\hat{\boldsymbol{x}}_2$ for completeness. We have in fact assumed their existence throughout the derivations. Notice that $\hat{\boldsymbol{q}}_1 = (\hat{\boldsymbol{x}}_1 + \hat{\boldsymbol{x}}_2)/\sqrt{2}$ and $\hat{\boldsymbol{q}}_1 = (\hat{\boldsymbol{x}}_1 - \hat{\boldsymbol{x}}_2)/\sqrt{2}$ which (aside from a multiplicative constant) is the same as our *clever solution* in the previous section.

- solve for the initial values $q_i(0) = Q_i$ by taking the dot products $\hat{\boldsymbol{q}}_i \cdot \boldsymbol{x}(0)$

$$\begin{aligned}
Q_1 &= \hat{\boldsymbol{q}}_1 \cdot \boldsymbol{x}(0) \\
&= \frac{1}{\sqrt{2}}(1 \;\; 1)\cdot\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}(x_1(0) + x_2(0)) \\
Q_2 &= \hat{\boldsymbol{q}}_2 \cdot \boldsymbol{x}(0) \\
&= \frac{1}{\sqrt{2}}(1 \;\; -1)\cdot\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}(x_1(0) - x_2(0))
\end{aligned}$$

- write down the final solution vector $\boldsymbol{x}(t)$, and obtain the individual functions $x_i(t)$ by taking the dot products $\hat{\boldsymbol{x}}_i \cdot \boldsymbol{x}(t)$

Our final solution is

$$\boldsymbol{x}(t) = \frac{x_1(0) + x_2(0)}{\sqrt{2}}\cos(\sqrt{k/m}t)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{x_1(0) - x_2(0)}{\sqrt{2}}\cos(\sqrt{3k/m}t)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{3.6}$$

To obtain the individual block motions,

$$\begin{aligned}
x_1(t) &= \hat{\boldsymbol{x}}_1 \cdot \boldsymbol{x}(t) \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \boldsymbol{x}(t) \\
&= \frac{x_1(0) + x_2(0)}{2}\cos(\sqrt{k/m}t) + \frac{x_1(0) - x_2(0)}{2}\cos(\sqrt{3k/m}t) && (3.7) \\
x_2(t) &= \hat{\boldsymbol{x}}_2 \cdot \boldsymbol{x}(t) \\
&= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \boldsymbol{x}(t) \\
&= \frac{x_1(0) + x_2(0)}{2}\cos(\sqrt{k/m}t) - \frac{x_1(0) - x_2(0)}{2}\cos(\sqrt{3k/m}t) && (3.8) \\
& && (3.9)
\end{aligned}$$

which the reader will immediately recognize Equations 3.7 and 3.8 as the same solution found in Equation 3.5 obtained by Method 1.

### 3.3 Summary of the Coupled Example

**Method 1**

- write down the differential equations $F_i(x_1, x2, ...) = m_i \ddot{x}_i$ for all blocks $i$
- make a clever substitution $q_j = f(x_1, x2, ...)$ to replace all $x_j$ and obtain new differential equations $m_i q_i = -\omega_i^2 q_i$
- solve for $q_i(t)$ for all $i$. using the solutions earlier, $q_i(t) = Q_i \cos(\omega_i t)$, each $q_i$ oscillating with a single frequency $\omega_i$. Use the initial conditions to obtain the unknown coefficients
- use the clever substitution to restore the equations to ones involving $x_i$.
- identify the physical significance of the solution and make sure that it matches one's intuition

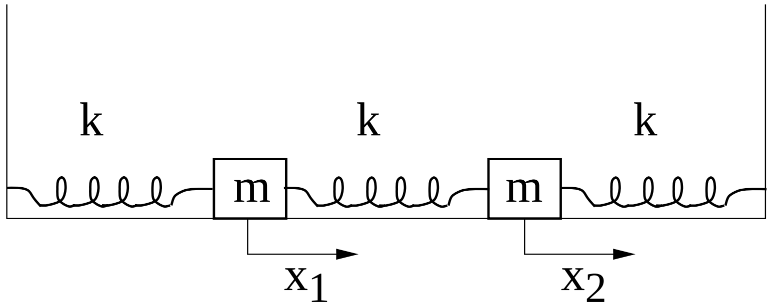
**Method 2**

- write down the differential equations $F_i(x_1, x2, ...) = m_i \ddot{x}_i$ for all blocks $i$
- place the equations in matrix notation: $\boldsymbol{M}\ddot{\boldsymbol{x}} = \boldsymbol{\Omega}\boldsymbol{x}$
- solve the matrix equation $[\boldsymbol{\Omega} - \boldsymbol{M}(-\omega^2)]\boldsymbol{q} = 0$ for all possible values $\omega$ and all possible vectors $\boldsymbol{q}$ which satisfy the equation. This is done by
  - · solving $\det|\boldsymbol{\Omega} - \boldsymbol{M}(-\omega^2)| = 0$ gives the possible values for $\omega$
  - · solve the matrix equation $[\boldsymbol{\Omega} - \boldsymbol{M}(-\omega^2)]\boldsymbol{q} = 0$ component-wise, to obtain a valid vector $\boldsymbol{q}$ for each valid $\omega$ value
- use superposition to restore the equations to ones involving $x_i$. This is done by
  - · write down the solution vector $\boldsymbol{x}(t)$ in terms of the vectors $\hat{\boldsymbol{q}}_i$
  - · solve for the initial values $q_i(0) = Q_i$ by taking the dot products $\hat{\boldsymbol{q}}_i \cdot \boldsymbol{x}(0)$
  - · write down the final solution vector $\boldsymbol{x}(t)$, and obtain the individual functions $x_i(t)$ by taking the dot products $\hat{\boldsymbol{x}}_i \cdot \boldsymbol{x}(t)$
- identify the physical significance of the solution and make sure that it matches one's intuition

## A Hints

### A.1 Hints on getting the force signs correct

There is a nice method for getting the signs correct when writing down the force on the two blocks in the following system.




Let's look at the force on block 1. The force from the spring connecting block 1 to the wall only depends on the value of $x_1$. Its value is then $F = -kx_1$. The force from the spring connecting block 1 and block 2 depends on both of positions $x_1$ and $x_2$, or more specifically on their difference $(x_1 - x_2)$, or $(x_2 - x_1)$. Getting the sign correct could be a little tricky, but it is crucial to get correct.

Now, imagine holding block 2 still and moving block 1 in a *positive* $x$ direction (shown by the arrow in the diagram as $x_1$ increasing). The force on block 1 is back, towards the wall, or in the $-x$ direction. If we hold block 1 still, and move block 2 in the *positive* $x$ direction, then the force on block 1 is in the $+x$ direction. Thus the force on block 1 from the spring connecting the blocks must me $F = k(x_2 - x_1)$, and the total force is

$$F_1 = -kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2$$

Let's look at the force on block 2. Again, the force from the spring connecting block 2 to the wall only depends on the value of $x_2$: $F = -kx_2$. The force from the spring connecting block 2 and block 1 can be figured out in the same way as we did for block 1, but we don't need to go through that extra work. We know that it must be of the opposite sign! Therefore the total force on block 2 is

$$F_2 = -kx_2 - k(x_2 - x_1) = kx_1 - 2kx_2$$

Imagining the direction of the force as one changes one particular block will help make sure that the signs are correct, and using the Newton's second law will save one some work.