Determining the Geometry of a Three-sided Fair Coin

Exploring the Probability of a Coin Landing on its Edge

B. Blais

Department of Science and Technology, Bryant University
Institute For Brain and Neural Systems, Brown University

October 2005

Introduction

Solutions
- Previous Solutions
- Proposed Solutions

Comparisons

Conclusions

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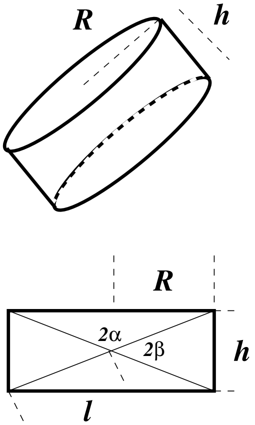
Coin



Defintions

- $\eta \equiv h/R$
- $l = \sqrt{R^2 + (h/2)^2}$
- $\beta \equiv \text{atan}\left(\frac{h}{2R}\right) = \text{atan}(\eta/2)$

The Question

What is the probability, p_{edge}, for the coin to land on the edge, as a function of the radius, R, and the height, h?

Restricted version of the question for a “fair” coin:

What values of h and R yield $p_{\text{edge}}(h, R) = 1/3$?

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Surface Area

- Probability proportional to the surface area

$$p_{\text{edge}}(h,R) = \frac{2\pi R h}{2\pi R^2 + 2\pi R h}$$
$$p_{\text{edge}}(\eta) = \frac{\eta}{1+\eta}$$

For the fair coin we obtain

$$\frac{\eta}{1+\eta} = \frac{1}{3}$$
$$\eta = \frac{1}{2}$$

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For the fair coin we obtain

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\eta &= \frac{1}{2}
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Cross-Sectional Length

- Probability proportional to the cross-sectional length

$$p_{\text{edge}}(h, R) = \frac{h}{2(2R) + h}$$

$$p_{\text{edge}}(\eta) = \frac{\eta}{4 + \eta}$$

For the fair coin we obtain

$$p_{\text{edge}}(\eta) = \frac{1}{3} = \frac{\eta}{4 + \eta}$$

$$\eta = 2$$

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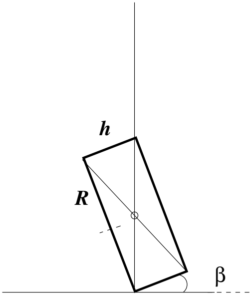
Center of Mass



β determines tip-over direction

$$\beta \equiv \text{atan}\left(\eta/2\right)$$

- if $0 < \theta < \beta$ then the coin will land on edge
- if $\beta < \theta < \pi/2$ then the coin will land on the heads

$$p_{\text{edge}}(h,R) = \frac{\beta}{\pi/2} \equiv p_e$$

For a fair coin we obtain

$$p_{\text{edge}}(h,R) = \frac{1}{3} = \frac{\beta}{\pi/2} \Rightarrow \alpha = 60^\circ$$

$$h/R = \frac{2}{\sqrt{3}} \approx 1.155$$

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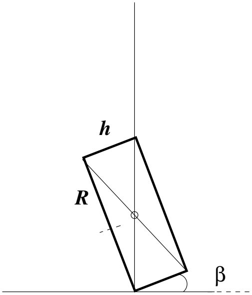
Center of Mass



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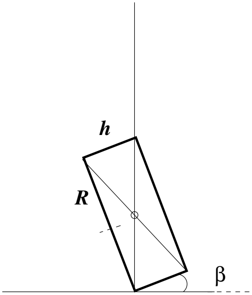

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Solid Angle(Mosteller, 1987; Pegg, 1997)

- Probability proportional to the solid angle occupied by each side extended onto the unit sphere

$$
\begin{aligned}
\text{edge area subtended} &= \int_0^{2\pi} d\phi \int_{\pi/2-\beta}^{\pi/2+\beta} \sin\theta d\theta d\phi \\
&= 4\pi \sin\beta \\
p_{\text{edge}}(\eta) &= \sin\beta \\
&= \frac{\eta}{\sqrt{\eta^2+4}}
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Dynamic Model(Murray and Teare, 1993)

- Simulate dynamic model of coin in 1-D with bouncing
- Free motion

$$
\begin{aligned}
E &= Z(t) + \frac{1}{2}V^2(t) + \frac{1}{2}k^2\omega^2(t) \\
Z(t) &= Z_o + V_o t + \frac{1}{2}t^2 \\
V(t) &= V_o - t \\
\theta(t) &= \theta_o + \omega t
\end{aligned}
$$

- Impact

$$
\begin{aligned}
V' &= V - (1+\gamma)k^2 \frac{V + x\omega}{(k^2 + x^2)} \\
\omega' &= \omega - (1+\gamma)x \frac{V + x\omega}{(k^2 + x^2)}
\end{aligned}
$$

- Corner velocities $U' = -\gamma U$

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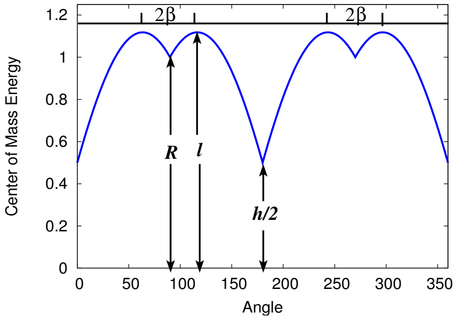
Simple Bounce Model (Blais)



- probability for an edge-inducive fall (COM):

$$p_e = \frac{\beta}{\pi/2}$$

- coin energy after n bounces:

$$E_n = E_o \gamma^n$$

- number of bounces, given initial and final energy:

$$n = \log(E_n/E_o)/\log\gamma + 1 = \log\left(\frac{l-h/2}{l-R}\right)/\log\gamma + 1$$

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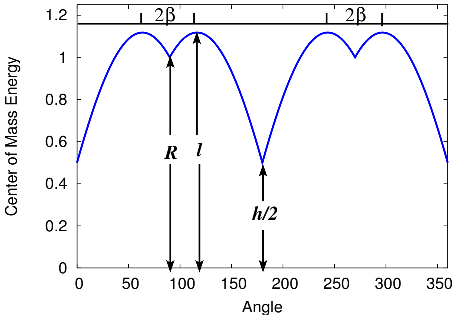

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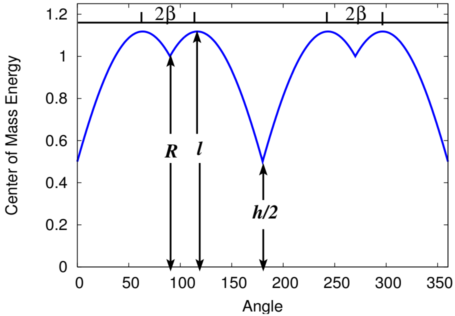


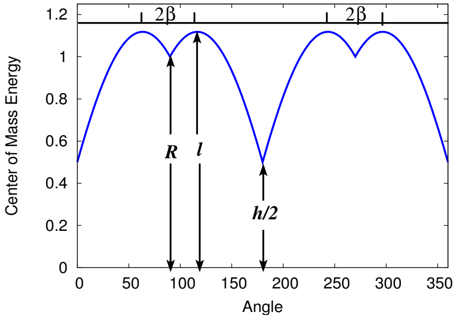
- number of bounces

$$n = = \log\left(\frac{l - h/2}{l - R}\right) / \log\gamma + 1$$

- probability of finally landing on edge

$$p_{\text{edge}} = p_e^n$$

Annealing Bounce Model (Blais)

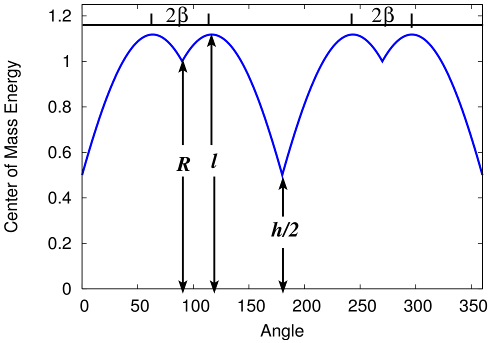


- Well of depth m and coin KE ϵ, escape probability

$$F(\epsilon) \approx \begin{cases} 0 & \epsilon < 0 \\ \sqrt{\frac{\epsilon}{m}} & 0 < \epsilon < m \\ 1 & \epsilon > m \end{cases}$$

- F_e and F_h as the wells for edge and heads, with depths $m_e = l - R$ and $m_h = l - h/2$

Annealing Bounce Model (Blais)



- Recursive equation

$$
\begin{aligned}
p_{\text{edge}}(i) &= p_e(1 - F_e(E_i)) + \\
&\quad p_e F_e(E_i) p_{\text{edge}}(i+1) + \\
&\quad p_h F_h(E_i) p_{\text{edge}}(i+1)
\end{aligned}
$$

- on edge, not escape
- on edge, escape, on edge next
- on heads, escape, on edge next

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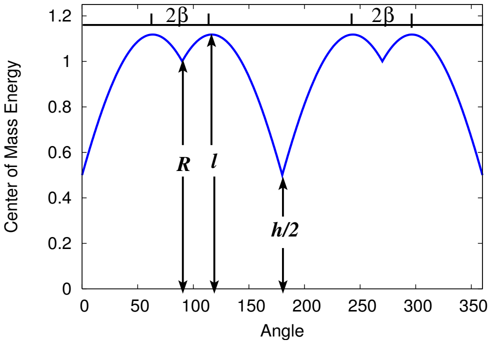

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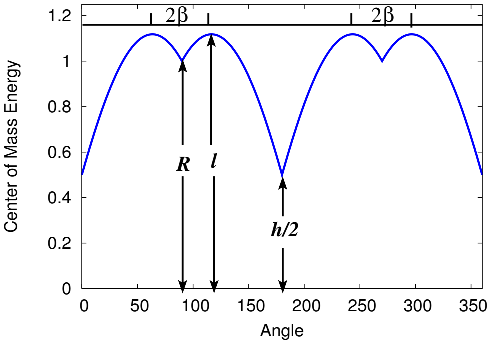

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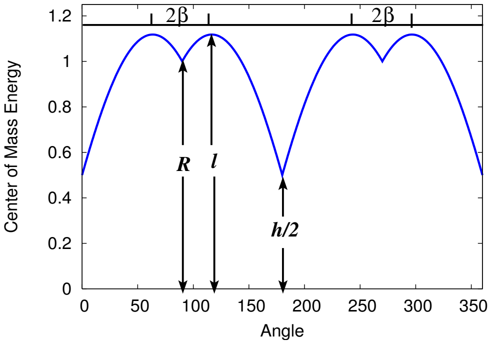

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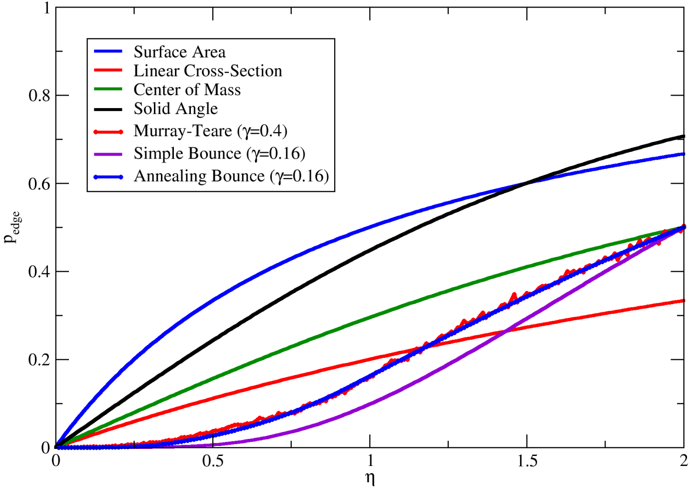
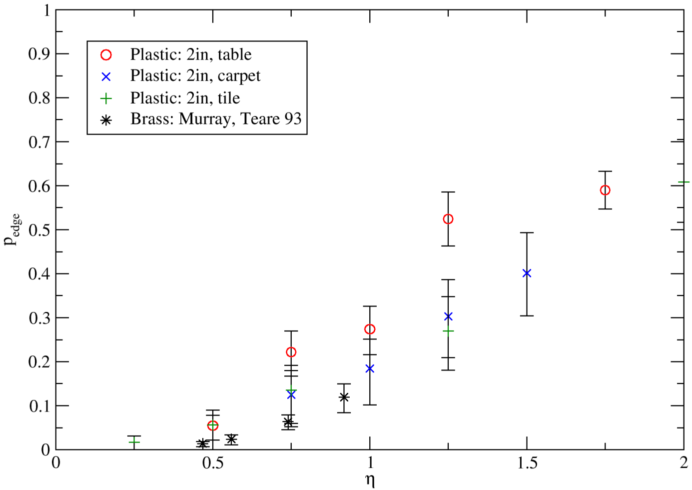
Models



Surface Area
Linear Cross-Section
Center of Mass
Solid Angle
Murray-Teare (γ=0.4)
Simple Bounce (γ=0.16)
Annealing Bounce (γ=0.16)
p_{edge}
η
0
0.2
0.4
0.6
0.8
1
0
0.5
1
1.5
2

Data



Plastic: 2in, table
Plastic: 2in, carpet
Plastic: 2in, tile
Brass: Murray, Teare 93
p_{edge}
η
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.5
1
1.5
2

Bayesian Model Comparison

- Probability of the model given the data

$$P(M_i|D,I) = \frac{P(D|M_i,I)P(M_i|I)}{P(D|I)}$$

- Compare two models

$$\frac{P(M_i|D,I)}{P(M_j|D,I)} = \frac{P(D|M_i,I)P(M_i|I)}{P(D|M_j,I)P(M_j|I)}$$

- If all of the models are *a priori* equally likely

$$\log P(M_i|D,I) - \log P(M_j|D,I) = \log P(D|M_i,I) - \log P(D|M_j,I)$$

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Comparisons

Model	Log Likelihood
Area	-1229.1
Length	-233.71
Center of Mass	-397.64
Murray-Teare (fixed γ)	-30.668
Simple Bounce (fixed γ)	-69.730
Simple Bounce (marginalized γ)	-32.415
Annealing (fixed γ)	-31.906

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