# Oscillations in a BCM Neuron: Perturbation Theory

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## Objective

The purpose of this study is to determine oscillatory solutions to the BCM neuron equations. I wish to determine how these solutions depend on the internal parameters of the neuron and on the neuron's environment.

## 1 The Model

The BCM neuron model can be easily summarized by the following three equations.

$$c = \boldsymbol{m} \cdot \boldsymbol{d} \tag{1.1}$$
$$\dot{\boldsymbol{m}} = \eta c(c-\theta)\boldsymbol{d} \tag{1.2}$$
$$\dot{\theta} = \frac{1}{\tau}(c^2 - \theta) \tag{1.3}$$

Equation 1.1 gives the activity $c$ as a function of the input vector $\boldsymbol{d}$ and the weight vector $\boldsymbol{m}$. Equation 1.2 is the *learning rule*, which determines how the weight vector changes in time. The learning rate, $\eta$, is one of the internal neuron parameters. Finally, Equation 1.3 determines how the threshold $\theta$ changes in time. Another form of this equation, which I will use in most of the analytic work, is

$$\theta = \frac{1}{\tau}\int_{-\infty}^{t} c^2(t')e^{-(t-t')/\tau}dt' \tag{1.4}$$

which shows clearly that the internal parameter $\tau$ is a *memory* of a running average of the squared activity, decayed by an exponential.

Oscillatory solutions occur when, say, the *memory constant* $\tau$ is large, so $\theta$ changes slowly. This allows the weight vector $\boldsymbol{m}$ to increase in value for a long time, and only start to decrease when $\theta$ has attained the value of $c$. After this occurs, $\boldsymbol{m}$ decreases, again for a long time while $\theta$ slowly reacts to the change. In this way the values of $\boldsymbol{m}$ and $\theta$ oscillate, perhaps without ever converging.

In this study I will look at a model where the $\boldsymbol{m}$ (and likewise $\boldsymbol{d}$) are one dimensional. The value of $d$ will be constant in time and we will look at solutions for $m$ about its fixed point.

## 2 Simple Oscillatory Behavior

### 2.1 The Calculations

The fixed point for $m$ is easy to find. It is determined by setting $\dot{m} = 0$.

$$\begin{aligned}
d &\equiv \text{(constant)} \\
m \equiv m_o \text{ (constant)} &\Rightarrow \dot{m} = 0 \\
&\Rightarrow c = m \cdot d = m_o d = \theta = \frac{1}{\tau}\int_{-\infty}^{t} m_o^2 d^2 e^{-(t-t')/\tau}dt' = m_o^2 d^2 \\
&\Rightarrow m_o d = 1 \Rightarrow m_o = \frac{1}{d}
\end{aligned} \tag{2.5}$$

The next step, now that we have the fixed point, is to add an oscillatory perturbation to the constant $m_o$ solution given in Equation 2.5.

We start with

$$\begin{aligned} c &= md \\ \dot{m} &= \eta c(c-\theta)d \\ \theta &= \frac{1}{\tau}\int_{-\infty}^{t} c^2(t')e^{-(t-t')/\tau}dt' \end{aligned}$$

and assume

$$m = m_o + m_1 \sin\omega t$$

where $m_1 \ll 1$ (i.e. we drop terms of order $m_1^2$ and higher) and $m_o = 1/d$.

It follows, after some algebra,

$$\begin{aligned} \theta &= \frac{1}{\tau}\int_{-\infty}^{t} (m_o + m_1\sin\omega t')^2 d^2 e^{-(t-t')/\tau}dt' \\ &\overset{m_1\ll 1}{\approx} [1] + \left[\frac{2m_1 d}{1+\omega^2\tau^2}\right]\sin(\omega t) + \left[\frac{-2m_1 d\omega\tau}{1+\omega^2\tau^2}\right]\cos(\omega t) \\ \dot{m} - \eta c(c-\theta)d &= 0 \\ &\overset{m_1\ll 1}{\approx} \left[\frac{m_1\eta d^2(1-\omega^2\tau^2)}{1+\omega^2\tau^2}\right]\sin(\omega t) + \left[\frac{m_1\omega(1+\omega^2\tau^2-2\eta d^2\tau)}{1+\omega^2\tau^2}\right]\cos(\omega t) = 0 \end{aligned}$$

which implies

$$\omega = \pm\frac{1}{\tau} \tag{2.6}$$

$$\frac{1}{\tau} = d^2\eta \tag{2.7}$$

### 2.2 The Interpretation

Equations 2.6 and 2.7 specify the criterion for *simple oscillations*. It has been shown that if the parameters $\tau$, $\eta$, and $d$ do not hold to Equation 2.7 then the oscillations are either damped out or they behave in a distinctly non-sinusoidal way. If Equation 2.7 is true, then the frequency is given quite closely by Equation 2.6.

## 3 Damped Oscillatory Behavior

### 3.1 The Calculations

The next obvious step is to assume a damped oscillatory solution for $m$ and proceed in the same way. This poses some immediate difficulties, however. The first difficulty is that the algebra is prohibitive. I have overcome this problem with the (grudging) help of the program Maple, which allows me to do symbolic manipulations. Even for Maple this problem is not totally straightforward. The other difficulty is that the integral for $\theta$ can no longer have a negative infinity limit, because the damping would blow up there. Unfortunately, carrying the integral from 0 to $t$ introduces more algebra. At this point I want to add that, even though Maple can do lengthy algebraic manipulations, one needs to hold its hand through them to avoid needless calculations. It has the tendency to give *extremely* long expressions which defy interpretation.

We start with the BCM equations, making sure to use a 0 lower limit for the $\theta$ integral

$$\begin{aligned} c &= md \\ \dot{m} &= \eta c(c-\theta)d \\ \theta &= \frac{1}{\tau}\int_0^t c^2(t')e^{-(t-t')/\tau}dt' + e^{-t/\tau}\theta_o \end{aligned}$$

and assume

$$m = m_o + m_1 e^{i\omega t}e^{-gt}$$

where $m_1 \ll 1$, $m_o = 1/d$ and $g$ is a positive damping constant.

$$\begin{aligned} \theta &= \frac{1}{\tau}\int_0^t (m_o + m_1 e^{i\omega t}e^{-gt})^2 d^2 e^{-(t-t')/\tau}dt' + e^{-t/\tau}\theta_o \\ &\overset{m_1\ll 1}{\approx} \frac{1}{\tau^2\omega^2+g^2\tau^2-2g\tau+1}\Big\{[2m_1 d\tau\omega]\, e^{-gt}\sin(\omega t) + [2m_1 d(1-g\tau)]\, e^{-gt}\cos(\omega t) + \\ &\quad [2m_1 d(g\tau-1)]\, e^{-t/\tau}\Big\} + [(\theta_o-1)]\, e^{-t/\tau} + \\ &\quad \frac{i}{\tau^2\omega^2+g^2\tau^2-2g\tau+1}\Big\{[2m_1 d\tau\omega]\, e^{-gt}\cos(\omega t) + [2m_1 d(1-g\tau)]\, e^{-gt}\sin(\omega t) + \\ &\quad [2m_1 d\tau\omega]\, e^{-t/\tau}\Big\} \end{aligned}$$

$$\begin{aligned} \dot{m} - \eta c(c-\theta)d &= 0 \\ &\overset{m_1\ll 1}{\approx} \left\{\frac{A_1}{B}e^{-gt}\sin(\omega t) + \frac{A_2}{B}e^{-gt}\cos(\omega t) + A_3 e^{-t/\tau}e^{-gt}\cos(\omega t) + \frac{A_4+A_5}{B}e^{-t/\tau}\right\} + \\ &\quad i\left\{\frac{-A_1}{B}e^{-gt}\cos(\omega t) + \frac{A_2}{B}e^{-gt}\sin(\omega t) + A_3 e^{-t/\tau}e^{-gt}\sin(\omega t) + \frac{A_6}{B}e^{-t/\tau}\right\} \end{aligned}$$

$$A_1 = \omega m_1(-(\tau^2\omega^2+g^2\tau^2-2g\tau+1) + 2\eta\tau d^2) = 0 \quad (3.8)$$

$$A_2 = m_1(-g\cdot(\tau^2\omega^2+g^2\tau^2-2g\tau+1) - d^2\eta(\tau^2\omega^2+g^2\tau^2+1)) = 0 \quad (3.9)$$

$$A_3 = d^2\eta m_1(\theta_o - 1) = 0 \quad (3.10)$$

$$A_4 = B\cdot(\theta_o-1)\eta d$$

$$A_5 = 2m_1\eta d^2(g\tau-1)$$

$$A_4 + A_5 = 0 \quad (3.11)$$

$$A_6 = 2\eta d^2 m_1\tau\omega \quad (3.12)$$

$$B = \tau^2\omega^2+g^2\tau^2-2g\tau+1 \neq 0 \quad (3.13)$$

What can we glean from these equations? Can we solve them at all, especially because Equations 3.11 and 3.12 are inconsistent with the entire approach (they imply that $m_1 = 0$)? We must introduce the assumption that $t \gg \tau$ to get rid of the $A_4 + A_5$ and $A_6$ troublesome terms. When we do this, we get a delightfully simple result.

$$\theta_o = 1 \quad (3.14)$$

$$\omega = \pm\frac{1}{2}\frac{\sqrt{6\tau\eta d^2 - 1 - (\tau\eta d^2)^2}}{\tau} \quad (3.15)$$

$$g = \frac{1}{2}\frac{(1-\tau\eta d^2)}{\tau} \quad (3.16)$$

## 3.2 The Interpretation

The meaning of Equation 3.14 is that the perturbation model assumed is most valid for $\theta_o = 1$. In the approximation $t \gg \tau$, the initial condition $\theta_o$ has little effect, so this result is not extremely useful.

Equations 3.15 and 3.16, however, have very important implications. These equations first of all suggest that the important aspects of the theory can be described using two parameters: $\tau$ and $\tau\eta d^2$ (which I will define as $\alpha$). In these new parameters, Equations 3.15 and 3.16 look like:

$$\omega = \pm\frac{1}{2}\frac{\sqrt{6\alpha - 1 - \alpha^2}}{\tau} \quad (3.17)$$

$$g = \frac{1}{2}\frac{(1-\alpha)}{\tau} \quad (3.18)$$

This simple appearance of the equations lets us divide the parameter space into several regions, based on the sign of $\omega$ and $g$. The simulations give us nice interpretations for these regions.

- **Region A:** $6\alpha - 1 - \alpha^2 \leq 0$

  In this region the frequency is either zero or pure imaginary. Simulations show that this results in the convergence of the neuron. Therefore when $\alpha \leq 3 - 2\sqrt{2}$ we get convergence.

- **Region B:** $6\alpha - 1 - \alpha^2 > 0$ and $\alpha < 1$ $(g > 0)$

  In this region we get simple, damped oscillations. The simulations and Equations 3.17 and 3.18 are in good agreement in this region.

- **Region C:** $\alpha > 1$ $(g < 0)$

  In this region we expect out theory to break down. If the damping constant is negative, then we get exponentially growing solutions, and our long-time approximation falls apart. Simulations show that the oscillations become increasingly nonlinear as $\alpha$ gets larger.