

Brian Blais' Homemade Guide to Error Analysis ¹

This is an informal guide to error analysis, which hopefully should clear up some of the more confusing ideas involved with interpreting scientific data. At the end of the guide is a summary of the formulae used in the guide, written in the standard mathematical notation.

1 The Big Picture

The problem of error analysis is one which presents itself in any experimental setting. One measures some quantities in an experiment, and through calculation obtains a number which hopefully describes some important facet of Nature. From a series of measured numbers one should be able to divine a ‘rule of Nature’ and gain a true understanding about what is going on around us everyday. Unfortunately, what one has to face is the fact that any measurement is imprecise; there are no perfect experiments.

The entire purpose of error analysis is to answer the question: *how much can one believe the result of an experiment?*

We are able to derive laws of Nature from the repetitive occurrence of results. But what is this repetition? Rarely do we reach precisely the same results in a series of trials. To use the criteria of repeatability, we must determine what “the same” means. What is similar enough? We will refer to the amount of similarity between measurements as the measure of the *repeatability* of an experiment.

Without quantitative limits on its repeatability, a result of an experiment is *totally meaningless*.

We will assume that Nature is working under some perfect rules which we are measuring imperfectly. Under this assumption, we will *expect* imperfect values. Our goal then will be to quantify how far off we expect these imperfect values are from the “true” perfect value of Nature. This should give us a measure of the repeatability of the experiment. Notice that *error* seen in this way is not a bad thing, but rather a measure of the limitations of both measurement itself and the assumptions we’ve made about Nature.

2 Introduction to the Problem

We will start with a fictitious land called Glopper, which has 5 cities: Nop, Pop, Mop, Bop, and Fop. We wish to discuss the temperatures in these cities on a chilly September day, and use our knowledge of error analysis to deal with the data taken. We find that on this day, at the same time, scientists from each of the five cities measure the temperature (T) of that city. The data looks like this:

City name	T (°C)
Nop	10.8
Pop	9.9
Mop	7.5
Bop	6.7
Fop	13.3

Table 1: Temperatures in the land of Glopper

¹For External Use Only.

Average Value (The Mean)

The first thing we wish to do is to attempt to find the “true” value of the temperature in the land of Glopper. We do this by taking an *average* with which most of you are familiar. To calculate the average, all we need to do is add up all of the values and divide by the total number of values (5 in this case, because we have 5 cities).

$$\begin{aligned}\bar{T} &= \frac{1}{5}(10.8^{\circ}C + 9.9^{\circ}C + 7.5^{\circ}C + 6.7^{\circ}C + 13.3^{\circ}C) \\ &= 9.6^{\circ}C\end{aligned}$$

When several measurements of the same thing are taken we *expect* there to be deviations between the measurements, but we expect the *average* value to remain constant no matter who performs the experiment. Thus, we can with some confidence call the average value, from the experiment, the “true” value, from Nature.

Percent Error

Now that we have measured the value of $9.6^{\circ}C$ for the mean temperature in Glopper, how can we determine whether to believe our result? One way we might think of is to use a standard *percent error* calculation (see Equation 8.2). Unfortunately, in this example, we don’t really have an accepted value of the temperature. If we somehow knew some laws of nature to calculate the temperature given the specific time and day in Glopper then we could use that theoretical value as the accepted value. Otherwise, we might have access to the weather records in Glopper, and compare our measured value to other measured values of the temperature on that time and day. In this way the accepted value with which we wish to compare our data can be theoretically or experimentally determined, and will thus be obtained differently for each problem you encounter. Let’s say that we know that on this day in the past the land of Glopper is, on average, $10.2^{\circ}C$, so we can calculate the *percent error*. The sign of the percent error only tells us whether the measured value is greater than or less than the accepted value.

$$\begin{aligned}\text{Percent Error} &= \frac{9.6^{\circ}C - 10.2^{\circ}C}{10.2^{\circ}C} \times 100 \\ &= -5.9\%\end{aligned}$$

This kind of analysis can work well only if we have a meaningful accepted value for comparison. In this case it is unsatisfying because we know that weather changes from year to year. A comparison with previous years tells us whether we have an especially cold year, or hot year, or even if we made some serious math error and got nonphysical results, but it tells us little about whether our measurements are valid or how accurate our experiment was. For this we need to look for consistency *within* the data, and use more sophisticated techniques.

3 Consistency within the Data

Standard Deviation

What if, instead of trying to find some external accepted value to compare our data with, we decide to compare our data to its own average value? What information could we possibly get from this approach? If we were to compare deviation of each measured value from the average value, we could get a quantitative grasp of the consistency of the measured data. All we need to do is take each value, subtract it from the mean of the data (giving us the *deviation* from the mean, and average all of these deviations in order to get a measure of consistency. For instance, if our data were extremely consistent, then each measured value would be very close

to the average value, so the *average deviation* would be low. If the data were inconsistent, then each value would lie far away from the average value, yielding a high average deviation. ‘Consistent data’ is another way of saying ‘data from a repeatable experiment’.

We must remind ourselves that we want to calculate consistency with a set of data, and therefore we must treat values above the mean in the same way we treat values below the mean. A value of 9.7°C deviates from the mean of 9.6°C just as much as 9.5°C does. We must, then, look at the *squares* of the differences of the measured values from the average value, not just the differences alone, because the *square* of a difference gives the same value whether the measured value is above or below the mean.

Mathematicians define some convenient functions for the consistency of data. We will only be using the *standard deviation*² and the *mean squared deviation*. The formulae are explicitly shown in Equations 8.3 and 8.4, but they merely represent what we have discussed before: squares of the differences between the data points and the average value.

The *mean squared deviation* is just the average of all of the squared differences described above, calculated in the same way as an average of values (see Equation 8.1). The *standard deviation* can be simply calculated, using Equation 8.4. The factor of $\sqrt{1/(N-1)}$ in the standard deviation equation ensures that one cannot calculate the error for a single data point (one would be dividing by zero). It is also unwise to calculate the standard deviation for sets of data with less than 4 or 5 values, because inaccuracies arise and our goal of repeatability would be compromised. Now the *standard deviation* of the temperatures in Table 1 is

$$\begin{aligned}\bar{T} &= 9.6^{\circ}\text{C} \\ \text{MS} &= \frac{1}{5}((9.6 - 10.8)^2 + (9.6 - 9.9)^2 + (9.6 - 7.5)^2 + (9.6 - 6.7)^2 + (9.6 - 13.3)^2) \\ &= 5.6^{\circ}\text{C}^2 \\ \text{SDoM} &= \sqrt{5.6^{\circ}\text{C}^2/(5-1)} \\ &= 1.2^{\circ}\text{C}\end{aligned}$$

The standard deviation is a measure of the consistency of the data set itself, thus the measurement in this experiment can be written as $\bar{T} \pm S.D.$, or in this case $9.6^{\circ}\text{C} \pm 1.2^{\circ}\text{C}$. Some people call this number the *error* in the measurement, but the word ‘error’ is too vague to be used without a qualifier, so I will use the term *total error* for the standard deviation, when it is used in this way. It refers to the fact that the data consistency that standard deviation measures includes all factors which influence it: systematic error, human error, random error, etc.

Significant Figures

Now that we have taken some measurements, calculating the value to be $9.6^{\circ}\text{C} \pm 1.2^{\circ}\text{C}$, we ask ourselves whether we expect this much consistency. To answer this question we must know how accurately we could have possibly measured the temperatures. If, for instance, our measuring apparatus were accurate to 0.1°C then we would have to seriously question our methods, or assumptions, when faced with the value 1.2°C for the standard deviation of the mean. Since we have not been told how accurate the measuring device is, how can we proceed? The answer lies in the significant figures of the data.

There is a convention in the sciences that whenever a measurement is made, the scientist guesses to one tenth of the finest mark on the measuring device. If one is measuring with a meter stick, which has marks at every millimeter, then one guesses to one tenth of a millimeter. Because of this, given a set of data, the

²Technically what I call the *standard deviation* is truly the *standard deviation of the mean*. I do this to make the language a little easier to follow, without giving up the quantitative correctness of the procedure. For an explanation of the difference between the *standard deviation* and the *standard deviation of the mean* see Taylor, J. R., *An Introduction to Error Analysis* or any other detailed books on error analysis.

reader can tell what the accuracy of the measuring device is without being told specifically what the device is. A length measurement of $0.2543m$ is accurate to $\pm 0.001m$. It is common for a beginning scientist to read a device, and drop off any extra zeros. Extra zeros at the end actually give information about the accuracy of the device. If one measures an object to be $0.2540m$ in length with one measuring device (ie. a meter stick), and then measures the same object with vernier calipers, one might get a length of 0.25400 . In this case, the extra zeros should *not* be dropped, because they say to the reader what the accuracy of the measurement actually is.

With the data in Table 1 we can see immediately that the scientists could measure temperature to $\pm 1^\circ C$. This corresponds well to the standard deviation calculated at $1.2^\circ C$. Notice that in all of the calculations above, each answer was rounded to the number of significant figures of the data. When there are more than one set of numbers in the data table, the calculated numbers are rounded to the *least* accurate data values (so the 22 digits from the sine function on one's calculator are not all significant in a calculation).

4 Testing some assumptions

Limitations of the Standard Deviation

Up until now we have used the standard deviation as a measure of the total error of a set of measurements (or the consistency of the measurements). It would stand to reason that, if we were to increase the accuracy of the thermometer that the scientists were using, the standard deviation would decrease, ie. the total error would be less, or the consistency of the data would be better. It turns out that we have some data taken by other (more careful) scientists on the same day and at the same time as the other scientists. This is shown in Table 2. Below are the calculations of the mean and standard deviation as well. One should also notice that the accuracy of the new measurements is to $\pm 0.1^\circ C$.

City name	T ($^\circ C$)
Nop	10.89
Pop	9.86
Mop	7.14
Bop	6.42
Fop	13.63

Table 2: Accurate Temperatures in the land of Gloppe

$$\begin{aligned}
 \bar{T} &= \frac{1}{5}(10.89^\circ C + 9.86^\circ C + 7.14^\circ C + 6.42^\circ C + 13.63^\circ C) \\
 &= 9.59^\circ C \\
 M.S. &= \frac{1}{5}((9.59 - 10.89)^2 + (9.59 - 9.86)^2 + (9.59 - 7.14)^2 + (9.59 - 6.42)^2 + (9.59 - 13.63)^2) \\
 &= 6.83^\circ C^2 \\
 S.D. &= \sqrt{6.83^\circ C^2 / (5 - 1)} \\
 &= 1.31^\circ C
 \end{aligned}$$

What has happened here?! We increased the accuracy of the experiment by a factor of ten, and our standard deviation went up! Because the new measurements were done to an accuracy of $\pm 0.1^\circ C$, our *expected error* is $\pm 0.1^\circ C$, but our *total error* (the standard deviation) is more than 10 times that! Either the new scientists introduced some new error into the experiment or our use or interpretation of the standard deviation has to change. The standard deviation is obviously not measuring what we want it to measure, namely the consistency of the data.

The biggest assumption that we have made in dealing with the standard deviation of our data, is that the data can be represented in the form $\bar{T} \pm S.D.$. We have assumed that, due to the laws of Nature, the temperatures are constant across the land of Glopper. The standard deviation works, assuming that we are measuring a *constant*, and that the average value over several measurements is giving us that constant. What if we have oversimplified our ideas about the laws of Nature (as we often do), and the temperatures are not constant? What if the positions of the cities geographically affect the temperature, or perhaps their altitude? One can easily see that our notion of the average value loses meaning. We must now ask ourselves again, how can we determine whether our experiment was a success, even given these extra variables? Can we salvage our ideas about standard deviation?

Finding a New Constant of Nature

In order to salvage the standard deviation, we must find some quantity that is related to the temperature, but is truly constant. Our problem arose in the following way. If our temperature measurements are not accurate, we might not pick up the difference in temperature caused by the various altitudes of the cities, so all of the land of Glopper would look approximately the same temperature. When we increased our accuracy, the effects of different altitudes became apparent, and increased our error (we could no longer think of temperature as a constant). What we can do is state that temperature (T) is a function of the altitude (h). We know that, as the altitude increases, the temperature decreases so we might want to try a function of the form³

$$T = \frac{\alpha}{h^2} \quad (4.1)$$

where α is some constant to be measured by experiment.

Now that we have a temperature as a function of altitude, if we obtain measurements of the altitudes of each city from the Glopper scientists, we can calculate the quantity α , which is presumably constant. We can compare how accurate our calculations of α are by using the new temperature data and the old temperature data and seeing which set of data is more consistent (ie. which set of data leads to a lower standard deviation of α). We have changed our focus, now, away from measuring the temperature (which we know is not constant in the land) to measuring α (which we presume to be constant).

City name	h (m)
Nop	940.1
Pop	984.8
Mop	1157.2
Bop	1233.3
Fop	836.4

Table 3: Altitudes in the land of Glopper

³This function is for demonstration purposes only, and is not intended to accurately model real data.

City Name	$T_{old} \text{ } ^\circ C$	$\alpha_{old} m^2 \text{ } ^\circ C$	$T_{new} \text{ } ^\circ C$	$\alpha_{new} m^2 \text{ } ^\circ C$
Nop	10.8	$9.54 \cdot 10^6$	10.89	$9.624 \cdot 10^6$
Pop	9.9	$9.6 \cdot 10^6$	9.86	$9.563 \cdot 10^6$
Mop	7.5	$1.0 \cdot 10^7$	7.14	$9.561 \cdot 10^6$
Bop	6.7	$1.0 \cdot 10^7$	6.42	$9.765 \cdot 10^6$
Fop	13.3	$9.30 \cdot 10^6$	13.63	$9.535 \cdot 10^6$

Table 4: Old and New values of the constant $\alpha = T \cdot h^2$

$$\begin{aligned} \overline{\alpha_{old}} &= 9.7 \cdot 10^6 m^2 \text{ } ^\circ C \\ S.D. \alpha_{old} &= 1.4 \cdot 10^5 m^2 \text{ } ^\circ C \\ \overline{\alpha_{new}} &= 9.610 \cdot 10^6 m^2 \text{ } ^\circ C \\ S.D. \alpha_{new} &= 4.151 \cdot 10^4 m^2 \text{ } ^\circ C \end{aligned}$$

Now we can see that the standard deviation is working as it should: the more accurate experiment has a smaller total error. We must remember that the standard deviation really has meaning only for measurements of true constants, and can lead to erroneous results if used improperly.

We can now quantify the total error in data, using the standard deviation, and we have seen when the average can and cannot be a useful quantity. The final question we must answer is how much error does one *expect*, given the specific apparatus that one is using (ignoring any human error)?

5 Expected Error

In the previous examples it was clear what the expected error of a given measurement was, just by inspecting the significant figures. T_{new} has an expected error of $\pm 0.1^\circ C$, and likewise the altitude h has an expected error of $\pm 1m$. Note that the *expected error* is just another word for the accuracy of the device. Now what is the expected error in a calculated quantity, like α ? How accurately do we *expect* to measure α , just given the accuracy of the devices we had to measure the quantities T and h ?

The error in the devices is not the only thing to consider here. Let's say we have a measuring device that measures length to $\pm 1m$, and we were to measure the width of a human hair with it. One would expect that error in that measurement would be really high, because the width of a human hair is so small. Now, if we were to be able to measure the distance between here and the next country to $\pm 1m$, one would immediately say that that is an extremely accurate measurement! Somehow both the precision of the measuring device *and* the quantity actually measured are important in considering how much error we expect in a measurement. In this way, we look at the quantity $\left(\frac{\Delta h}{\bar{h}}\right)^2$ in considering the expected error in a measurement of distance (height in our case). Δh is the precision of the measuring device ($1m$), and \bar{h} is just the average value of the measurements of h .

Let's get back to our question of how accurately we expect to measure α , which is calculated from other measured values. Equations 8.5-8.8 gives us the form for calculating the expected error of a calculated quantity. For instance, α in the equation $\alpha = T \cdot h^2$ is a product of 2 numbers (T and h^2). In mathematical language, α is of the form $Z = XY$, where Z is α , X is T and Y is h^2 . In that way, $\Delta X = \Delta T$ and $\Delta Y = \Delta(h^2)$, and we can then use Equation 8.7 to calculate $\Delta \alpha$. Remember that ΔT is the precision with which we have measured that value T , therefore $\Delta T = 0.1^\circ C$.

Now what is $\Delta(h^2)$? Normally it would be the precision with which we have measured the value h^2 , but we *didn't* measure h^2 . We measured h ! We notice that h^2 is *also* a product of 2 numbers (h and h), so $\Delta(h^2)$ has to then be calculated in much the same way that $\Delta\alpha$ was calculated, using Equation 8.7. We can now calculate $\Delta\alpha$ in terms of our measured quantities, T and h , and the precision we measured them with, ΔT and Δh . We must remember that $\Delta\alpha$ is how accurately we *expect* to measure α , just given the accuracy of the devices used to measure it. The calculations are shown below.

$$\begin{aligned}
 T &= \bar{T} \pm 0.1^\circ C \\
 h &= \bar{h} \pm 1m \\
 \frac{\Delta Z}{Z} &= \sqrt{\left(\frac{\Delta X}{X}\right)^2 + \left(\frac{\Delta Y}{Y}\right)^2} \\
 \frac{\Delta\alpha}{\alpha} &= \sqrt{\left(\frac{\Delta T}{T}\right)^2 + \left(\frac{\Delta(h^2)}{h^2}\right)^2} \\
 &= \sqrt{\left(\frac{\Delta T}{T}\right)^2 + \left(\frac{\Delta h}{h}\right)^2 + \left(\frac{\Delta h}{h}\right)^2} \\
 \frac{\Delta\alpha}{9.610 \cdot 10^6 m^2 \circ C} &= \sqrt{\left(\frac{0.1^\circ C}{9.59^\circ C}\right)^2 + \left(\frac{1m}{1030m}\right)^2 + \left(\frac{1m}{1030m}\right)^2} \\
 &= 0.033 \\
 \Delta\alpha &= 0.0330 \cdot 9.610 \cdot 10^6 m^2 \circ C \\
 &= 3.172 \cdot 10^5 m^2 \circ C \\
 \alpha &= 9.610 \cdot 10^6 m^2 \circ C \pm 3.172 \cdot 10^5 m^2 \circ C
 \end{aligned}$$

The value of our expected error is actually larger than the standard deviation, for this case. The scientists of Gloppler must have gotten some exceptionally consistent data on that day!

General Expected Error (Optional)

Many times the equations for a quantity we are studying cannot be put into a simple sum or product form. In order to deal with these we can use a little calculus and derive the equations we need (like equations 8.5-8.8).

To start, let's say that we are measuring a quantity T , which depends on several variables, say u, v , and w . What we want is to figure out how an expected error in the measurement of the three variables affects the expected error of the calculated quantity T . If we change u by Δu then the change in T will be

$$\Delta T(u, \Delta u) \equiv \Delta T_u = \frac{\partial T}{\partial u} \Delta u \quad (5.2)$$

This is just the slope of the function T , as a function of u (keeping all other variables constant), times the change in u thus giving the change in T . Similarly we can define ΔT_v and ΔT_w , the changes in T due to changes in v and w respectively. Now that we have the various changes in T , we construct a little box in the 3-D space of u, v , and w where the diagonal length gives us the expected error in T .

$$\Delta T = \sqrt{(\Delta T_u)^2 + (\Delta T_v)^2 + (\Delta T_w)^2} \quad (5.3)$$

Now that we have the math, we can go through an actual example. Earlier we found the expected error in the constant α as a function of T and h , using the product and sum rules. Let's do the same example using calculus, first finding how changes in T and h individually change α , and then using Equation 5.3 to get the

actual expected error.

$$\begin{aligned}
 \alpha &= Th^2 \\
 \Delta\alpha_T &= \frac{\partial\alpha}{\partial T}\Delta T = h^2\Delta T \\
 \Delta\alpha_h &= \frac{\partial\alpha}{\partial h}\Delta h = 2Th\Delta h \\
 \Delta\alpha &= \sqrt{(\Delta\alpha_T)^2 + (\Delta\alpha_h)^2} \\
 &= \sqrt{h^4(\Delta T)^2 + 4T^2h^2(\Delta h)^2} \\
 &= Th^2\sqrt{\left(\frac{\Delta T}{T}\right)^2 + \left(\frac{2\Delta h}{h}\right)^2} \\
 &= \alpha\sqrt{\left(\frac{\Delta T}{T}\right)^2 + \left(\frac{2\Delta h}{h}\right)^2}
 \end{aligned}$$

If you compare this result with the earlier result, you will notice immediately that we seem to be off by a factor of 2 dealing with the expected error in h . What we have done in this section is correct, and the other section has a slight problem due to the fact that the product $h \cdot h$ is not a product of 2 *independent* variables, but this distinction will probably not make much of a difference in the calculation. Usually, in fact almost invariably, the expected error will be significantly smaller than the standard deviation because the standard deviation includes *all* types of error, not merely the device limitations. This helps justify not being completely rigorous earlier, because we are after semi-quantitative results, where we can locate some of our sources of error and determine how much they might affect the overall error. What, then, are the other types of errors that creep into an experiment and make one's standard deviation high?

6 Other forms of Error

There is a tendency for the beginning scientist to look at error, and say that there were problems in the equipment, or there was human error which caused the error in the experiment. These errors are called *random errors*, and are difficult to quantify. It is difficult to say that, due to human error, we expect to be off by $\pm 1m$ in a length measurement because humans are inherently unpredictable (or at least very difficult to predict). There is one saving grace however: human error, and other forms of random error usually do not play a significant role in the error of an experiment. It is all too easy to shove the error off to human error and ignore the problem altogether but this gives one no measure of the believability of the measurement, which invalidates the entire experiment. What then can we attribute the error to, if not to human sloppiness or poor equipment?

By far the largest source of error in an experiment is a poor assumption. The nice and tidy equations one finds in theory, oftentimes do not accurately depict the experimental situation. In our example, the first scientists used an equation which *assumed* a constant temperature, and got results which were not bad. The second scientists, under the same assumption, got terrible results and found that they had to change their assumptions about the problem, and even change the problem itself, to get meaningful and predictable results. One should always look at equations being used, and write down the assumptions that are used in obtaining the equations. In this way, one can face the data and see which assumptions are valid, and which ones are not. One can also propose better assumptions, which might lead to better agreement with the data, and thus to a better understanding of the problem.

7 Conclusion

What we have seen here is a problem worked out from start to finish, exploring many of the aspects of error analysis and the science of measurement. All of the methods shown were trying to quantify how much we can trust an experiment. An experiment is *completely meaningless* unless one can convince others why the result should be believed. Quantifying repeatability in an experiment is the way to establish how much the result can be believed. To accomplish this we defined three types of error:

- *Percent error*, which is useful only when there is some accepted value to which one can compare one's data. The accepted value in the percent error could be a theoretically calculated value, or a constant which has been measured to some standard degree of accuracy. This establishes believability by comparing a result to results of the past.
- *Standard deviation*, which is a measure of the total error of a set of data (including human error and random error), and can also be thought of as a measure of the consistency of the data within itself. We noticed that the meaning of the standard deviation fell apart when dealing with data that was affected by other variables in the experiment. In those cases one has to find a formula which relates those variables, and obtain a new constant which can be measured and analyzed. We also found that the standard deviation was less meaningful for small sets of data. The standard deviation establishes believability by quantifying how repeatable a result is from one measurement to another.
- *Expected error*, which is a measure of the expected accuracy of the devices used, or of a calculated quantity, given the accuracy of the measuring devices needed for the calculation. This error tends to be much smaller than the others, due to the fact that it is *only* considering device accuracy. The expected error quantifies the credibility of the devices and measurement methods used in an experiment, and thus is another measure of the experiment's believability.

Behind all of this, we have to look at the assumptions we make about the laws of Nature and see if we can trust these assumptions or whether they can account for the errors mentioned above. In this way, the theory and experiment play a game of tag: the theory makes assumptions about Nature, and the experiment measures the natural quantities, forcing one to reexamine and reform the assumptions, which then can be tested.

Finally, we realize that the example displayed here is only an example. Each problem in the lab is slightly different, and should be handled individually. The skills described are general, however, and they should get one through most lab situations.⁴

⁴No animals were harmed in the production of this guide.

8 Summary of Key Formulae

We label measurements as $X_1, X_2, X_3, \dots, X_N$.

Average (Mean)

$$\begin{aligned}\bar{X} &= \frac{1}{N}(X_1 + X_2 + X_3 + \dots + X_N) \\ &= \frac{1}{N} \sum_{i=1}^N X_i\end{aligned}\tag{8.1}$$

Percent Error

$$\text{Percent Error} = \frac{\text{Value}_{\text{measured}} - \text{Value}_{\text{accepted}}}{\text{Value}_{\text{accepted}}} \times 100\tag{8.2}$$

Mean Squared Deviation and Standard Deviation⁵

$$\begin{aligned}\text{M.S.} &= \frac{1}{N}((\bar{X} - X_1)^2 + (\bar{X} - X_2)^2 + (\bar{X} - X_3)^2 + \dots + (\bar{X} - X_N)^2) \\ &= \frac{1}{N} \sum_{i=1}^N (\bar{X} - X_i)^2\end{aligned}\tag{8.3}$$

$$\text{S.D.} = \sqrt{\text{M.S.}/(N - 1)}\tag{8.4}$$

Expected Error

The **expected error** (ΔZ) of any value (Z) made up of products or sums of other values ($X \pm \Delta X$ and $Y \pm \Delta Y$):

$$\text{For } Z = X + Y\tag{8.5}$$

$$\Delta Z = \sqrt{(\Delta X)^2 + (\Delta Y)^2}$$

$$\text{For } Z = X - Y\tag{8.6}$$

$$\Delta Z = \sqrt{(\Delta X)^2 + (\Delta Y)^2}$$

$$\text{For } Z = XY\tag{8.7}$$

$$\frac{\Delta Z}{Z} = \sqrt{\left(\frac{\Delta X}{X}\right)^2 + \left(\frac{\Delta Y}{Y}\right)^2}$$

$$\text{For } Z = X/Y\tag{8.8}$$

$$\frac{\Delta Z}{Z} = \sqrt{\left(\frac{\Delta X}{X}\right)^2 + \left(\frac{\Delta Y}{Y}\right)^2}$$

⁵The S.D. defined here is properly known as Standard Deviation of the Mean. See Footnote 2