Fourier Diagonalization of Circular Matrices

Circular matrices creep into a lot of problems where one has interactions between similar elements in a system. The *interaction matrix* is usually only a function of the relative difference between the elements. Because these matrices are so common, it is good to find analytical tools which make dealing with them easier. Here we present a method whereby one can diagonalize a circular matrix in Fourier space, making easier analysis possible.

We start with a defintion of a Fourier transformation matrix

$$\text{define}: \quad U_{jk} \equiv \frac{1}{\sqrt{N}} e^{i2\pi jk/N}$$

and note the following theorem

Theorem 1 *U is a unitary transformation matrix.*
Proof

$$\begin{aligned}
(U^\dagger U)_{\alpha\beta} &= \frac{1}{N}\sum_{j=1}^{N} e^{-i2\pi j\alpha/N} e^{i2\pi j\beta/N} = \frac{1}{N}\sum_{j=1}^{N} e^{i2\pi j(\beta-\alpha)/N} \\
\text{define}: \ y &\equiv e^{i2\pi j(\beta-\alpha)/N} \\
y^N &= \left(e^{i2\pi j(\beta-\alpha)/N}\right)^N = e^{i2\pi j(\beta-\alpha)} \\
&= 1 \ \ \beta-\alpha \text{ is an integer)} \\
(U^\dagger U)_{\alpha\beta} &= \frac{1}{N}\sum_{j=1}^{N} y^j = \frac{y}{N}(1+y+y^2+\cdots+y^{N-1}) \\
&= \begin{cases} \frac{y}{N}\left(\frac{1}{1-y} - y^N\frac{1}{1-y}\right) = \frac{y}{N}\left(\frac{1-y^N}{1-y}\right) & \text{for } |y|<1 \ (\alpha\neq\beta) \\ \frac{1}{N}N = 1 & \text{for } |y|=1 \ (\alpha=\beta) \end{cases} \\
&= \begin{cases} 0 & \text{for } \alpha\neq\beta \\ 1 & \text{for } \alpha=\beta \end{cases} \\
(U^\dagger U)_{\alpha\beta} &= \delta_{\alpha\beta}
\end{aligned}$$

Therefore, U is a unitary transformation matrix. □

Now we define a circular matrix M as a matrix which is only a function of the relative difference of its indices.

$$\begin{aligned}
\text{define}: \ \text{circular matrix } M_{jk} &= M_{jk}(|j-k|) \\
&= M(j-k) = M(j-k\pm N) \quad (j,k=1,\ldots,N)
\end{aligned}$$

Next we transform the problem into Fourier space by using the transformation matrix U.

$$\begin{aligned}
\text{define}: \ D &\equiv U^\dagger M U \\
D_{\alpha\beta} &= \frac{1}{N}\sum_{j,k=1}^{N} e^{-i2\pi j\alpha/N} M_{jk} e^{i2\pi k\beta/N} \\
&= \frac{1}{N}\sum_{j,k=1}^{N} M_{jk} e^{-i2\pi(j\alpha-k\beta)/N} = \frac{1}{N}\sum_{j,k=1}^{N} M(j-k) e^{-i2\pi(j-k)\alpha/N - i2\pi k(\alpha-\beta)/N} \\
&= \frac{1}{N}\sum_{k=1}^{N} e^{-i2\pi k(\alpha-\beta)/N} \sum_{j=1}^{N} M(j-k) e^{-i2\pi(j-k)\alpha/N}
\end{aligned}$$

It is now time to propose a theorem which will simplify the expression for D.

Theorem 2 *The expression* $\sum_{j=1}^{N} M(j-k)e^{-i2\pi(j-k)\alpha/N}$ *is independant of* k.
Proof

$$\begin{aligned}
\text{define}:\ l &\equiv j-k \quad (k=1,2,\ldots,N \quad l=1-k,2-k,\ldots,N-k)\\
\left(\sum_{j,k=1}^{N}\right) &= \left(\sum_{k=1}^{N}\sum_{l=1-k}^{N-k}\right)\\
\sum_{j=1}^{N} M(j-k)e^{-i2\pi(j-k)\alpha/N} &= \sum_{l=1-k}^{N-k} M(l)e^{-i2\pi\alpha l/N}\\
&= \sum_{l=1-k}^{0} M(l)e^{-i2\pi\alpha l/N} + \sum_{l=1}^{N-k} M(l)e^{-i2\pi\alpha l/N}\\
\text{circular matrix assumption} \Rightarrow\ &= \sum_{l=N-k+1}^{N} M(l)e^{-i2\pi\alpha(l-N)/N} + \sum_{l=1}^{N-k} M(l)e^{-i2\pi\alpha l/N}\\
&= \sum_{l=N-k+1}^{N} M(l)e^{-i2\pi\alpha l/N}e^{-i2\pi\alpha} + \sum_{l=1}^{N-k} M(l)e^{-i2\pi\alpha l/N}\\
(e^{-i2\pi\alpha}=1) \quad \Rightarrow\ &= \sum_{l=1}^{N} M(l)e^{-i2\pi\alpha l/N}
\end{aligned}$$

Therefore,

$$\sum_{j=1}^{N} M(j-k)e^{-i2\pi(j-k)\alpha/N} = \sum_{l=1}^{N} M(l)e^{-i2\pi\alpha l/N}$$

is independant of k. □

Finally we make the convenient definition

$$\text{define}:\quad \lambda_\alpha \equiv \sum_{l=1}^{N} M(l)e^{-i2\pi\alpha l/N}$$

which then simplifies D to

$$\begin{aligned}
D_{\alpha\beta} &= \lambda_\alpha \frac{1}{N}\sum_{k=1}^{N} e^{-i2\pi k(\alpha-\beta)/N}\\
&= \lambda_\alpha (U^\dagger U)_{\alpha\beta}\\
&= \lambda_\alpha \delta_{\alpha\beta}
\end{aligned}$$