Brian Blais' Quick Homemade Guide to those Pesky **D**'s and **H**'s

This guide is going to try to serve as a quick summary of Electric and Magnetic fields in matter. I assume that you can look up the details in Purcell. I will keep with CGS units throughout, because that is what Purcell uses.

When one looks at electric fields in materials, one is immediately confronted with the fact that the electric field *polarizes* the material. This means that the material becomes a large collection of electric dipoles (each atom becoming a dipole). This immediately leads us to say that charged objects create electric field partially by the extra charge we have slapped on (*free* charges), and partially due to the *bound* charges making up the dipoles in the material. The bound charge density is thus somehow related to the dipole moment density (also called the polarization). The specific relation is given below. Here we write down Maxwell's equations, and note that the charge density has a *free* part and a *bound* part, where the bound part is related to the dipole moment density of the material. We represent it mathematically like

$$\nabla \cdot \mathbf{E} = 4\pi\rho = 4\pi(\rho_{free} + \rho_{bound}) = 4\pi(\rho_{free} + (-\nabla \cdot \mathbf{P}))$$

Because the problem of finding $\mathbf{E}$ is now quite difficult (with the potentially complicated $\mathbf{P}$ in the equation), we simplify things by getting the free charge density on one side, and everything else on the other.

$$\nabla \cdot (\mathbf{E} + 4\pi\mathbf{P}) = 4\pi\rho_{free} \equiv \nabla \cdot \mathbf{D}$$

We haven't done any new physics here, we have just rewritten things such that the divergence of the field $\mathbf{E} + 4\pi\mathbf{P}$ depends only on the *extra* charge we slap on, and we are left with something much like the original Gauss' Law.

We can do the same thing for the magnetic field as well, only with the curl of $\mathbf{B}$. We start with Maxwell's equations, introduce a *free* and *bound* current, and then bring the hard stuff on one side leaving the final expression simple. Remember that the *free* current is much like the *free* charge, in that it refers to the extra stuff we have control over (and is not related to the behavior of the atoms in the material). We, like in the electric field case, associate the bound current with the magnetic dipole moment density in a way that you can look up.

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} = \frac{4\pi}{c}(\mathbf{J}_{free} + \mathbf{J}_{bound}) = \frac{4\pi}{c}(\mathbf{J}_{free} + c\nabla \times \mathbf{M})$$

$$\nabla \times (\mathbf{B} - 4\pi\mathbf{M}) = \frac{4\pi}{c}\mathbf{J}_{free} \equiv \nabla \times \mathbf{H}$$

Putting it together, one finally obtains Maxwell's equations in matter. Here $\mathbf{D}$ is just the total electric field minus all of the dipole effects from the polarization (or bound charge). Likewise $\mathbf{H}$ is just the total magnetic field minus all of the bound current effects.

$$\nabla \cdot \mathbf{D} = 4\pi\rho_{free} \tag{1.1}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{1.2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.3}$$

$$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J}_{free} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} \tag{1.4}$$

Immediately one has to ask the following question: who cares? How has introducing all of this new notation helped us solve problems any easier? To demonstrate this, one has to go through an example. Essentially now that we have, say, Equation 1.1 in the form it is in, we can use the nifty tricks we used for Gauss' law on $\mathbf{D}$ instead of $\mathbf{E}$. For example, consider the system of a charged metal sphere with some material surrounding it (say, jello). From Gauss' law we can immediately write down $\mathbf{D} = \frac{Q}{r^2}\hat{r}$. How about the electric field inside a capacitor with jello in it? $\mathbf{D} = 4\pi\sigma$. And the magnetic field inside a wire of radius R and made of jello (which, of course, has a well known magnetization)? $\mathbf{H} = \frac{2Ir}{cR^2}\hat{\phi}$, as you can readily remind youself with the same problem with no magnetization.

Now that we have easy ways of obtaining $\mathbf{D}$ and $\mathbf{H}$, we still can't get the total fields $\mathbf{E}$ and $\mathbf{B}$ without knowing the (potentially complicated) polarization ($\mathbf{P}$) or magnetization ($\mathbf{M}$). Luckily we can *approximate* these by saying that the polarization is just a constant times the applied electric field, and similarly for the magnetization and the magnetic field.

$$\mathbf{P} = \chi_e \mathbf{E}$$

$$\mathbf{M} = \chi_m \mathbf{H}$$

Notice: it would've been nice if the magnetization were a constant times $\mathbf{B}$, but convention (and only that) dictates the above form. This make the remembering harder, but the physics is the same and the math is no more difficult. Plugging the above expressions into Maxwell's equations, we get (and this is just juggling constants, no new physics)

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \mathbf{E} + 4\pi\chi_e\mathbf{E} = (1 + 4\pi\chi_e)\mathbf{E} \equiv \epsilon\mathbf{E}$$

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M} = \mathbf{H} + 4\pi\chi_m\mathbf{H} = (1 + 4\pi\chi_m)\mathbf{H} \equiv \mu\mathbf{H}$$

Now this allows us to do two things. First thing it allows us to do is to immediately write down the full fields in the examples before, assuming of course that the polarization and magnetization of jello follows our little approximation. The field inside the jello surrounding the metal sphere then becomes simply $\mathbf{E} = \frac{Q}{\epsilon r^2}\hat{r}$, so then the polarization becomes $\mathbf{P} = \chi_e\mathbf{E} = \frac{\chi_e Q}{(1+4\pi\chi_e)r^2}\hat{r}$. Trying to find this polarization first would have been very difficult indeed! A similar procedure will work for the other examples. Notice, however, that Maxwell's equations aren't quite symmetric with repect to $\mathbf{E}$'s and $\mathbf{D}$'s (Equation 1.2), so the symmetry arguments we used here break down for more complicated systems (such as time varying systems) but work quite well for many simpler problems.

The second thing we can do with this, now that we have a recipe to get the polarization and magnetization, is use our knowledge of electric and magnetic dipoles to see how these materials interact. We know the field produced by a dipole, and the force and torque it experiences in external fields, both of which can now be used in describing these systems.